Accretional Evolution of a Planetesimal Swarm

2. The Terrestrial Zone

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We use our multi-zone simulation code (D. Spaute, S. Weidenschilling, D. R. Davis, and F. Marzari, Icarus 92, 147–164, 1991) to model numerically the accretion of a swarm of planetesimals in the region of the terrestrial planets. The hybrid code allows interactions between a continuum distribution of small bodies in a series of orbital zones and a population of large, discrete planetary embryos in individual orbits. Orbital eccentricities and inclinations evolve independently, and collisional and gravitational interactions among the embryos are treated stochastically by a Monte Carlo approach. The spatial resolution of our code allows modeling of the intermediate stage when particle-in-a-box methods lose validity due to nonuniformity in the planetesimal swarm. The simulations presented here bridge the gap between such early-stage models and N-body calculations of the final stage of planetary accretion. The code has been tested for a variety of assumptions for stirring of eccentricities and inclinations by gravitational perturbations and the presence or absence of damping by gas drag. Viscous stirring, which acts to increase relative velocities of bodies in crossing orbits, produces so-called “orderly” growth, with a power-law size distribution having most of the mass in the largest bodies. Addition of dynamical friction, which tends to equalize kinetic energies and damp the velocities of the more massive bodies, produces rapid “runaway” growth of a small number of embryos. Their later evolution is affected by distant perturbations between bodies in non-crossing orbits. Distant perturbations increase eccentricities while allowing inclinations to remain low, promoting collisions between embryos and reducing their tendency to become dynamically isolated. Growth is aided by orbital decay of smaller bodies due to gas drag, which prevents them from being stranded between orbits of the embryos.

We report results of a large-scale simulation of accretion in the region of terrestrial planets, employing 100 zones spanning the range 0.5 to 1.5 AU and spanning 10⁶ years of model time. The final masses of the largest bodies are several times larger than predicted by a simple analytic model of runaway growth, but a minimal-mass planetesimal swarm still yields smaller bodies, in more closely spaced orbits, than the actual terrestrial planets. Longer time scales, additional physical phenomena, and/or a more massive swarm may be needed to produce Earth-like planets.

I. INTRODUCTION

Most newly formed stars are encircled by disk-shaped nebulae of gas and dust that may be precursors of planetary systems. The processes by which such a nebula becomes
planets remains something of a mystery. We have only one example of a “normal” planetary system: our own. Some of its characteristics were presumably determined by initial conditions, e.g., the mass of the Sun and its nebular disk (Boss 1995); others may have been the outcome of stochastic events during the formation of the planets. We do not know the degree to which our Solar System is typical or only one of a wide range of possible outcomes. Dynamical constraints can rule out some combinations of planetary masses and orbits, but great diversity is allowed (Lissauer 1995). It has been suggested that the dominant feature (by mass) of our own system, i.e., giant planets such as Jupiter and Saturn, may be rare (Wetherill 1994); however, the recent discoveries of massive planets orbiting other stars (Mayor and Queloz 1995, Marcy and Butler 1996, Butler and Marcy 1996, Cochran et al. 1996) suggest they are relatively common. The configurations of these systems differ greatly from our own. Until other planetary systems are detected in numbers sufficient to reveal their true diversity, we must depend on the uncertain and incomplete methods of theory and numerical modeling.

There has developed a consensus version of a qualitative “standard” model of planetary formation (cf. Levy 1985, Wetherill 1990, Lissauer 1993). In this model, the terrestrial planets formed by collisional accumulation of a swarm of small bodies (planetesimals) of rocky composition, in orbit about the Sun. The cores of the giant planets accumulated in similar fashion, but the presence of ices at greater heliocentric distances allowed them to grow large enough to accrete gas from the nebula. There are some problems with this model, particularly regarding time scales; it is hard to explain how the cores of the giant planets could grow to the size of \( \sim 10 M_\oplus \) \((M_\oplus = \text{Earth's mass})\) required to accrete gas in the interval \( \sim 10^5 - 10^7\) years estimated for the lifetime of the gaseous nebula (Podosek and Cassen 1994). This process may require a nebular mass several times higher than the minimum value (Pollack et al. 1996) or additional phenomena, e.g., tidal torques (Ward and Hahn 1995), that have not yet been incorporated into the quantitative models of planetary formation.

There is less constraint on the accretion of the terrestrial planets, as they need not have formed before the nebular gas dissipated. Analytic estimates (Safronov 1969) suggested time scales of \( \sim 10^8 \) years to complete the accretion process. In the analytic model, planetary growth occurred as a competition among many bodies having a quasi-equilibrium collisional size distribution, i.e., a power law with most of the mass in the largest bodies. The process is controlled by a balance between coagulation in collisions and stirring of relative velocities (orbital eccentricities and inclinations) by gravitational perturbations. This scenario is often referred to as “orderly growth” (although it is orderly only in a mathematical sense; the final stage involves high-velocity collisions of planet-sized bodies).

Numerical modeling, which requires fewer simplifying assumptions, revealed another possible course of planetary growth. Greenberg et al. (1978) found rapid “runaway” growth of a small number of large planetesimals while most of the mass remained in small bodies. Wetherill and Stewart (1989) clarified this result, showing that it depended on “dynamical friction,” a tendency for gravitational interactions to produce equipartition of kinetic energy. This effect was implicit in the Greenberg et al. simulations (Kolvoord and Greenberg 1992), and explicit in the Wetherill and Stewart model. The more massive bodies have lower random velocities and hence larger collisional cross sections (augmented by their gravity). This feature was lacking in Safronov’s analytic stirring model, which was “positive definite”; i.e., stirring only acted to increase velocities.

When dynamical friction is included, runaway growth is characteristic of the earliest stage of accretion. As the large bodies formed in this way become more numerous and constitute a significant fraction of the total mass of the planetesimal swarm, their perturbations stir the smaller bodies and each other. Their relative advantage due to gravitational cross section diminishes, and runaway growth slows (Ida and Makino 1993). Also, runaway apparently cannot produce a final configuration of bodies in widely spaced orbits having long-term stability (Kokubo and Ida 1995). Thus, growth of planetesimals presumably involved a transition from initial runaway to a longer-lived stage of high-velocity collisions among large bodies, resembling the final stage of the orderly growth model. As described below, modeling the initial and final stages of planetary accretion is relatively tractable; the intermediate stage is more difficult.

The initial number of planetesimals in the terrestrial region was very large, perhaps \( \sim 10^{12}\) bodies if they were roughly kilometer sized. It is impossible to treat them as individual objects. Instead, the planetesimal swarm can be modeled by continuous distributions, e.g., of sizes and orbital parameters. The evolution of theses distributions can be calculated by a variety of approaches. Analytic theory (Safronov 1969) is limited by the need to make simplifying assumptions to render the problem tractable. An integro-differential coagulation equation can describe the evolution of the planetesimal size distribution (Ohhtsuki and Nakagawa 1988), but its validity is questionable for cases with runaway growth (Tanaka and Nakazawa 1994). The most direct approach involves numerical calculations of a system of interacting particles, solving for changes in their sizes and velocities (Greenberg et al. 1978, Hayakawa et al. 1989, Wetherill and Stewart 1989, 1993). All of these methods are limited by the explicit or implicit assumption that the swarm is uniform (in number density, size distributions, orbital eccentricities, and inclinations, etc.) over some region of solution in which the planetesimals interact.
This uniformity is essential for the validity of “particle-in-a-box” models. If the evolving swarm is to produce a planetary system, it must become nonuniform at some stage as the inevitable result of the emergence of a small number of large bodies, or planetary embryos. Their growth is the phenomenon of greatest interest, but unfortunately it renders invalid the original assumption of uniformity.

A number of phenomena tend to make the planetesimal swarm nonuniform as it evolves. The largest bodies in the swarm may dominate the regions near their orbits, causing the distribution of sizes and velocities of the smaller bodies to vary with location. Different regions can interact by collisions and gravitational scattering where orbits cross, by longer-range perturbations between bodies in non-crossing orbits (Weidenschilling 1989), or by resonant perturbations at selected locations (Weidenschilling and Davis 1985, Kary et al. 1993). Small bodies may be depleted near an embryo’s orbit as it accretes them; they may be replenished by diffusion of orbits in gravitational scattering (Nakagawa 1978), secular decay of orbits due to gas drag (Weidenschilling 1977a), or increasing eccentricities that allow crossing of more distant orbits. Massive embryos may migrate due to tidal torques from the gaseous disk nebula (Ward and Hahn 1995). Such processes limit the validity of the concept of a “feeding zone” containing the mass destined to accrete into a planet. Some effects of nonuniformity may be delayed or reduced by careful choice of “box” size and imposition of certain restrictions. For example, one may not allow collisions between bodies if they are so few that their mean orbital separation would exceed some threshold (Wetherill and Stewart 1993); this implicitly assumes some degree of uniformity in the spacing of their orbits. In any case, the assumptions of particle-in-a-box models must eventually break down.

Besides the problem of nonuniformity, another complication encountered in numerical modeling is the form of the size distribution. In the case of runaway growth, the large bodies may be too few to be represented by a continuous size distribution, while there are too many small bodies to be represented individually. At this stage the small bodies constitute most of the mass of the swarm and cannot be neglected. At a much later stage of evolution, most of the mass will be in a small number of large bodies. The final stage of accretion can then be simulated by Monte Carlo orbital calculations for a few hundred such bodies (e.g., Wetherill 1990), or even direct integration of their orbits; however, the starting conditions for such end-stage models must be assumed, because they have not been directly linked to the results of early-stage accretion models. The goal of our work is to close this gap between models of early and late stages of planetary formation.

II. THE MULTI-ZONE SIMULATION MODEL

To overcome the limits of early-stage particle-in-a-box accretion models, and to provide a linkage between them and end-stage N-body simulations, we have developed a hybrid multi-zone model for collisional and orbital evolution of a swarm of planetesimals. Many of its features were described by Spaute et al. (1991, hereafter Paper I; see also Spaute et al. 1985). The simulations are carried out for a series of zones of heliocentric distance. Planetesimals are assigned to zones on the basis of their orbital semimajor axis $a$. They have Keplerian orbital elements (eccentricity $e$ and inclination $i$) rather than a single “random velocity.” A unique feature of our code is its treatment of the size distribution of planetesimals. The smaller bodies are treated as part of a continuum size distribution, specified by the numbers of bodies in a series of logarithmic mass bins. Within each zone, the continuum bodies are assumed to have $e$’s and $i$’s that are uniformly distributed between minimum and maximum values that are in a fixed ratio. When accretion of continuum bodies produces objects larger than a specified mass, they are treated as discrete bodies in individual Keplarian orbits (their initial $a$, $e$, and $i$ are chosen randomly from the appropriate continuum bin that produces them). Discrete bodies interact with the continuum and each other; their masses and orbits evolve due to collisions and gravitational stirring. Bodies in different zones may also stir and/or collide with each other, and these interactions can transfer mass between zones. This feature is distinct from the “multiple zone” simulations of Nakagawa et al. (1983) and Ohtsuki et al. (1988), which were actually simultaneous particle-in-a-box calculations for separate zones that did not interact directly. In our code, zones need not be contiguous to interact; a body with sufficiently large eccentricity and/or gravitational cross section may encounter bodies belonging to many other zones.

We have made many improvements to the physical model described in Paper I. We summarize them briefly here and in more detail in Appendices A–E.

Collisions. The earlier version of the code allowed bodies to collide if their orbits overlapped, based on current (unperturbed) values of perihelion and aphelion distances. This is a reasonable approximation when relative velocities are high and radial excursions (~$ae$) are large compared with the gravitational cross section for collisions; however, this is not the case for large bodies experiencing runaway growth. Collisions are possible between bodies in orbits that do not overlap, if their unperturbed orbits can approach within a few times the radius of their mutual Hill sphere, defined as

$$R_{\text{H}} = \left[ \frac{m_1 + m_2}{3M_\odot} \right]^{1/3} \frac{a_1 + a_2}{2},$$

(1)

where $(m_1, a_1)$, $(m_2, a_2)$ are the masses and semimajor axes, and $M_\odot$ the solar mass. In the restricted three-body
problem, a small body of negligible mass can have close encounters or collide with a large body in a circular orbit if its orbit meets the condition

\[ \frac{3h^2}{4} - e_{\text{H}}^2 - i_{\text{H}}^2 < 9, \quad (2) \]

where \( b, e_{\text{H}}, \) and \( i_{\text{H}} \) are the difference in semimajor axes and the eccentricity and inclination of the small body's orbit, all scaled to the Hill radius (i.e., \( b = |a_2 - a_1|/R_{\text{H}} \), \( e_{\text{H}} = ae_{\text{rel}}R_{\text{H}} \), \( i_{\text{H}} = a \sin i/R_{\text{H}} \); cf. Greenzweig and Lissauer 1990). The quantity on the left-hand side is a version of the Jacobi constant, and is conserved through multiple gravitational encounters that change \( a, e, \) and \( i \) of the small body; however, this condition only allows (but does not require) long-term evolution to produce collisions. For initially circular, coplanar orbits, collisions are allowed if \( b < 2\sqrt{3} \), but in the first synodic period, only orbits with \( 1.8 \leq b \leq 2.6 \) make close approaches (Greenzweig and Lissauer 1990). Smaller values of \( b \) represent horseshoe- or Trojan-type orbits, while orbits with larger \( b \) pass by outside the large body's Hill sphere; subsequent synodic encounters may yield closer approaches after perturbations cause eccentricities to increase. For our purposes, computing the time-dependent collision rate, we need to decide whether close approaches can occur at any given instant, given in Appendix C of Wetherill and Stewart (1993), rather than the long-term possibility. There is no such criterion that is both simple and rigorous, especially as it also must apply to bodies of comparable mass, both of which may have eccentric orbits. Based on results from numerical integrations of many encounter trajectories by Ida (1990), Greenzweig and Lissauer (1990), and Ipatov (1994), we allow collisions to occur if the closest approach of two orbits (difference between perihelion of the outer orbit and aphelion of the inner) is less than \( 2.4R_{\text{H}} \). Bodies with larger separations may collide eventually, after gravitational stirring increases their eccentricities and allows closer approaches. The probability of collision is computed as described in Appendix A, excluding horseshoe- and Trojan-type orbits (Appendix C).

**Gravitational focusing and collisional cross-section.** The rate at which planetesimals encounter each other depends on their relative velocity, which has two elements: a random component due to their orbital eccentricities and inclinations, and a systematic velocity due to the variation of orbital velocity with distance from the Sun ("Keplerian shear"). The cross section for collision is enhanced over their geometric area by gravitational focusing. The rate of collisions is due to the combination of these effects. The collision rate increases with decreasing random velocities but reaches a limiting value when \( e \) and \( i \) are small enough that Keplerian shear dominates. The transition between these regimes occurs at relative velocities \( V_{\text{rel}} = 2V_{\text{H}} \), where \( V_{\text{H}} = \Omega R_{\text{H}} \), the local Kepler frequency times the Hill radius.

There have been many numerical and analytic investigations of the collision rate in the shear-dominated regime (Wetherill and Cox 1985, Ida and Nakazawa 1989, Nakazawa et al. 1989a, b, Greenzweig and Lissauer 1990, 1992). Their results have generally been expressed as a total accretion rate of planetesimals in a swarm of uniform surface density, integrated over all orbits that could collide with a protoplanetary embryo. For our calculations, we must consider subsets (individual zones of semimajor axis) of a swarm that will generally be nonuniform in surface density from one zone to another. For our purposes, the semianalytic model of Greenberg et al. (1991) is more readily adaptable to subsets of the swarm. We use their expressions to determine the rate of encounters and probabilities of collision between planetesimals in different zones and size bins.

**Gravitational stirring.** The description of our code in Paper I and the simulations described therein assumed a fixed ratio of inclination to eccentricity, \( i/e = 0.5 \), because the stirring model of Stewart and Wetherill (1988) was based on this assumption. More recently, Stewart has developed a new model for stirring rates of \( i \) and \( e \) taken separately, with arbitrary ratio of \( i/e \). We use the formulas given in Appendix C of Wetherill and Stewart (1993), with some modifications as described below. One can show (Appendix B) that Stewart's equations tend to produce \( i/e = 0.6 \) when stirring by bodies in crossing orbits dominates planetesimal velocities and the effect of more distant bodies in non-crossing orbits is small.

Wetherill and Stewart account for the gravitational stirring by objects in non-crossing orbits by adding a constant value (0.55) to the term \( \ln \Lambda \) in the coefficient of the stirring rate. This value was suggested by Weidenschilling (1989) to approximate the total effect of velocity perturbations, both in the orbital plane and perpendicular to it. Use of this constant is a reasonable approximation for fixed \( i/e \), as in Stewart and Wetherill's (1988) theory; however, distant perturbations are predominantly in the plane of the orbit, and \( i \) and \( e \) much more than \( i \). For stirring models that decouple \( e \) and \( i \) it would be a better approximation to add the constant 0.55 only to the coefficient for eccentricity (it should also be omitted from the dynamical friction terms).

In our own code, we do not use this approach, but rather evaluate the in-plane and out-of-plane stirring rates directly from the expression given by Weidenschilling (1989). For each pair of discrete bodies, their mutual perturbations are computed directly using their known orbital separations and inclinations.

Safronov (1969) and Wetherill and Stewart (1989) noted that because the Jacobi parameter is conserved, a single large planetary embryo would be relatively ineffective at
stirring a swarm of small bodies. They suggested that in such cases, the large body would be no more effective at viscous stirring than the second largest body that the small ones could encounter (this limitation does not apply to distant perturbations; cf. Greenzweig and Lissauer 1990). Wetherill and Stewart (1993) found that this effect was less important than they had assumed; in their simulations the eccentricities of the small bodies and spacings of embryos usually did not produce such isolation. We also find this to be the case. In our spatially resolved simulations the stirring algorithm always determines the largest and second-largest body that can interact with any subset of the stirred bodies, and scales their stirring rate accordingly.

Strictly speaking, the limit on stirring by an isolated embryo refers to velocities of the small bodies relative to the embryo itself; they may still acquire significant velocities relative to each other by scattering encounters with the embryo. Greenzweig and Lissauer (1992), Ida and Makino (1993), and Kokubo and Ida (1995) used N-body numerical simulations to study this phenomenon. They found significant stirring of small bodies in a region extending \( \pm 4\sqrt{3} R_H \) on either side of the embryo’s orbit. This region corresponds to the range for which we compute viscous stirring, and the region just beyond that is most strongly affected by distant perturbations. While our formalism is different, our stirring model is in good agreement with their results. It appears, however, that it is preferable to model the largest body as fully effective at stirring the smaller bodies in its vicinity (Appendix B). Our simulations reported here were carried out with the largest body’s stirring assumed no more effective than the second largest; however, this assumption has little if any effect on the outcomes. In most circumstances the smaller bodies have sufficiently large eccentricities to encounter more than one embryo, so the two largest bodies encountered are generally comparable in size. The exceptions are generally seen in the later stages of accretion near the edges of the swarm. Because we do not consider discrete bodies that would presumably form in a larger swarm beyond our modeled region, bodies in the edge zones may experience less stirring and have lower eccentricities. This effect is noticeable in some of our simulations.

Viscous stirring and dynamical friction as expressed by Stewart’s equations are collective properties of a system containing many particles that can be modeled as a continuum. As such, they may not be valid for describing the interactions between the largest bodies when they are sparsely distributed. We do not use these equations to model the mutual gravitational perturbations of the discrete bodies. Orbits that cross or approach within \( 2.4R_H \) are subject to close encounters that result in gravitational scattering. The encounter probability is computed in the same manner as the probability of collisions, but with a larger cross section (Appendix B4). The impact parameter and angles that define the encounter are chosen randomly, with distributions appropriate for the flattening of the system (Greenberg et al. 1991). We model each encounter as a two-body hyperbolic scattering event, and compute new orbits for the bodies from their velocity impulses. At low relative velocities the two-body approximation does not accurately describe single encounters, but is a good estimate of the averaged effect of many such events (Wetherill and Cox 1984). Dynamical friction is not explicit in this model, but we note that the less massive body undergoes the larger velocity deflection. One important aspect of this model that does not appear with the continuum equations is the possibility of sudden stochastic jumps in \( a, e, \) and \( i \). The instantaneous values of \( i, a, \) and \( e, \) and their ratio, may be very different from the “equilibrium” values predicted by the stirring equations.

Wetherill et al. (1996) have compared results of a continuum model, employing averaged stirring and collision rates, with a numerical integration of actual orbits. They found good agreement between the resulting distributions of masses, eccentricities, and inclinations for the two methods; however, the numerical integration yielded substantial fluctuations about the mean values at larger masses, when small numbers of bodies were represented. Their simulation was limited to 3000 bodies in a narrow zone, \( 0.04 \text{AU} \) wide, and growth in mass by a factor of 300. It is not clear when (or if) the continuum method of modeling ever breaks down so far as to affect the qualitative outcome of a simulation, but we expect stochastic effects to become increasingly important in later stages of accretion.

**Shepherding** Viscous stirring (i.e., close encounters between objects in crossing orbits) also leads to chaotic changes in semimajor axes, which cannot be easily evaluated. Distant encounters, however, produce systematic changes in orbital separation (semimajor axes) that accompany stirring of \( e \) and \( i \). These changes in \( a \) can be evaluated from conservation of the Jacobi integral; it is not necessary to assume that either body has negligible mass (Ida and Nakazawa 1989). In general, an increase in \( e \) and/or \( i \) by distant perturbations increases the separation of their semimajor axes (although their distance of closest approach decreases due to the larger \( e \)). Thus there is some tendency for pairs of discrete bodies to “repel” each other. The effect is more pronounced for small bodies stirred by large ones; this phenomenon, coupled with damping of eccentricities by collisions, gas drag, or dynamical friction, is responsible for “shepherding” of small bodies. For multiple large bodies embedded in a swarm of small ones, this effect can drive the embryos apart or bring them closer together, depending on their initial separation and distribution of small bodies (Kokubo and Ida 1995). Our code includes this evolution of semimajor axes in a manner consistent with velocity stirring by distant perturbations (Appendix B).
Horseshoe orbits. Two bodies in similar orbits may be protected from collisions or close approaches if they are in 1:1 resonance, i.e., Trojan- or horseshoe-type orbits. We use the criterion of Ida and Nakazawa (1989), which includes the bodies’ eccentricities as well as the difference in their semimajor axes (scaled with \( R_{H} \)). Two bodies in horseshoe orbits do not collide; a similar but more restrictive criterion (Appendix C) may prevent mutual encounters that could stir their velocities. Their semimajor axes experience a step change each synodic period, with the less massive body undergoing the larger change; in a many-body system these excursions in semi-major axis may actually destabilize the ensemble of orbits. In our simulations, a significant fraction (tens of percent) of discrete bodies may be in horseshoe-type resonances with another body at some stage of the swarm’s evolution. With rare exceptions, these resonances are temporary and are eventually broken by increases in eccentricity or encounters with a third body.

III. TEST SIMULATIONS

We have performed a series of simulations of accretion of a planetesimal swarm in the region of the terrestrial planets, to determine the sensitivity of the outcomes to different assumptions. We initially tested a variety of models for gravitational stirring, including a Safronov-type model without dynamical friction and three simulations with dynamical friction, with and without long-range perturbations and orbital decay due to gas drag. In these simulations, the region modeled is 0.3 AU in width and centered on a heliocentric distance of 1 AU; it extends from 0.85 to 1.15 AU in semimajor axis. The region is divided into 30 zones of width 0.01 AU. The initial population of planetesimals for the test simulations (Fig. 1) is chosen to match the starting conditions used by Wetherill and Stewart (1993), with a surface density of the swarm equal to 16.7 g cm\(^{-2}\) at 1 AU. We assume that the surface density varies as \( a^{-3/2} \) initially (number of bodies per zone \( \propto a^{-1/2} \)), giving a total mass of the swarm of \( 7.15 \times 10^{27} \) g = 1.2\( M_{\oplus} \) within the distance limits. The planetesimals have mass \( m = 4.8 \times 10^{18} \) g, density 3.0 g cm\(^{-3}\), and diameter 14.5 km. We have determined that, all else being equal, the outcomes of the simulations, i.e., final sizes and orbital spacings of the largest bodies, are insensitive to the initial size. We assume that the modeled region is embedded within a more extended swarm, and so impose reflective boundary conditions: whenever collisions or gravitational perturbations produce orbits with semimajor axes beyond these boundaries at 0.85 and 1.15 AU, they are “reflected” to new values within the boundaries at the same distance from the edges (this process simulates exchange of planetesimals with a more extended swarm). Bodies near the edges of the modeled region experience fewer collisions, as they lack neighbors on one side; this produces noticeable edge effects in some of the simulations. The threshold for creating discrete bodies from the continuum in these cases is taken to be \( M = 2 \times 10^{23} \) g, corresponding to bodies 500 km in diameter.

The code does not rigorously conserve angular momentum; however, a typical simulation without gas drag (which causes secular decay of orbits and loss of angular momentum) conserves angular momentum to better than one part in \( 10^3 \) over a model time of \( 10^6 \) years.

In these simulations, all collisions are assumed to result in accretion. Fragmentation requires a significantly greater number of mass bins to deal with bodies much smaller than the starting population and a correspondingly larger amount of computer time (roughly increasing as the square of the number of size bins). We will consider fragmentation in a future publication.

A. “Safronov” Case with “Positive Definite” Stirring

In the first case, we approximate the gravitational stirring model of Safronov (1969) by eliminating the dynamical friction expressions (Eqs. C7a, b of Wetherill and Stewart 1993). We assume no damping of velocities by gas drag and no stirring by distant perturbations (their effects are small compared with the viscous stirring terms in this case). Results are shown in Figs. 2 and 3.

The lack of dynamical friction produces eccentricities and inclinations that are essentially independent of planetesimal size. The “positive definite” stirring causes the mean \( e \) and \( i \) to increase monotonically; the random velocity is typically of the order of the escape velocity of the dominant (largest) bodies. The ratio \( \sin i/e \) remains near 0.6, as expected from the stirring model (Appendix B).

The innermost zones show more rapid growth initially due to the combination of higher surface density and shorter orbital periods. This effect is not very pronounced, as there is no runaway growth. The size distribution evolves into a power law of uniform slope, of the form \( dN \propto m^{-q} dm \), with \( q \approx 1.5 \). The shallow slope means that most of the mass is in the larger bodies, as would be expected from Safronov’s analytic theory. Growth is “orderly,” at least until the number of largest bodies becomes small enough after \( \sim 10^6 \) years for stochastic encounters and gravitational scattering to become important relative to viscous stirring. After a model time of \( 3 \times 10^6 \) years, about two-thirds of the total mass is in discrete bodies with a mean mass \( \sim 10^{25} \) g; the largest bodies are \( \approx 10^{26} \) g.

B. Adding Dynamical Friction; No Distant Perturbations

In the next case, we add the dynamical friction terms to the stirring equations. There is no stirring by distant perturbations; the stirring and dynamical friction are applied only to bodies in crossing orbits (i.e., approaching within 2.4\( R_{H} \)). Damping of \( e \) and \( i \) by gas drag is also
FIG. 1. Initial conditions used in the simulations. The values are plotted for logarithmic intervals of mass; each size bin spans a factor of 2 in mass. Zones of semimajor axis have uniform width 0.01 AU. At $t = 0$, the planetesimals have a uniform size (mass $4.8 \times 10^{18}$ g), occupying the smallest size bin in each zone. The number of planetesimals is proportional to $a^{-1/2}$, giving surface density $\propto a^{-3/2}$, mean eccentricity $9.7 \times 10^{-5}$, and $\sin i = 5.8 \times 10^{-5}$. The total mass is $1.2M_\oplus$ in a region of width 0.3 AU.

included, with gas density $1.18 \times 10^{-9}$ g cm$^{-3}$ at 1 AU; the surface density of gas is assumed proportional to the surface density of the planetesimal swarm. Because of the large initial sizes of the planetesimals ($\geq 14.5$ km diameter), the gas has little effect on their velocities or the results. In this simulation and in Case C, we allow gas drag to damp eccentricities and inclinations, but do not include secular decay of semimajor axes due to drag. The damping rate due to the gas is that which was used by Wetherill and Stewart (1989).

As the initial planetesimals accrete to form larger bodies, there is a strong tendency toward equipartition of random kinetic energy; i.e., eccentricities and inclinations are smaller for bodies of larger mass (Fig. 4). The low velocities of the more massive bodies ($m > 10^{21}$ g) cause them to have larger gravitational cross sections, promoting runaway growth as described by Wetherill and Stewart (1989). The size distribution has a shallower slope for these large bodies, unlike the simple power law seen in Case A. Growth is much more rapid than in the Safronov case, with the largest bodies reaching $\sim 10^{24}$ g after 10$^4$ years. Stochastic collisions among these bodies produce a small number of still larger embryos that grow fast enough to separate from the leading edge of the size distribution. These embryos grow to masses approaching $10^{27}$ g in 10$^5$ years.

Viscous stirring tends to produce a ratio $i/e \approx 0.6$. During the early, most rapid stage of runaway growth, $i/e$ increases temporarily for the larger bodies. We attribute this effect to the fact that our model assumes a distribution of $e$ and $i$ about the mean. The collision rate is higher for orbits with lower inclination; these are preferentially removed from a given size bin during accretion. There is a similar effect for low-eccentricity orbits, but it is not as strong; the net result is an increase in $i/e$ until the runaway growth slows down and viscous stirring has a chance to “catch up” with accretion. Then $i/e$ remains near the equilibrium value of 0.6 except for stochastic effects due to occasional scattering events between discrete bodies or collisions involving depleted continuum bins that contain only a few planetesimals.

In this simulation runaway growth proceeds rapidly for the first $\approx 5 \times 10^4$ year, at which time the largest bodies have reached masses of a few times $10^{26}$ g (less than half the mass of Mars). After this point, growth slows dramatically, and most of the embryos do not reach $10^{27}$ g even after times approaching $10^6$ years. The mass at which growth stalls is comparable to the total starting mass in the single zone of semimajor axis, but this appears to be coincidental. If a growing embryo swept up all matter within $2.4R_H$ of its orbit, for our choice of parameters its growth would stall at $5 \times 10^{26}$ g (Lissauer and Stewart 1993). Nonzero orbital eccentricities of the planetesimal would make more mass available.

The runaway bodies of mass $\sim 10^{26}$--$10^{27}$ g have low random velocities; as seen in Fig. 5, there is a strong tendency toward equipartition among the discrete bodies. The largest bodies are spaced fairly evenly at intervals of a few times $10^{-2}$ AU. At $10^6$ years there are eight bodies with $m > 10^{26}$ g. Their eccentricities are only $\sim 10^{-4}$, and they...
FIG. 2. Evolution of distributions of mass, eccentricity, and inclination for Case A, with no gas drag or dynamical friction. Growth is “orderly,” with the size distribution resembling a power law. Eccentricities and inclinations show little dependence on size at any given time. Several million years are required to produce bodies of mass \( \sim 10^{26} \) g.
previously, we evaluate distant perturbations explicitly for all interacting bodies rather than adding a constant to the stirring coefficient, and the contributions to $e$ and $i$ are determined separately.

In a well-mixed system in which crossing orbits are common, viscous stirring is much more effective than distant perturbations (Weidenschilling 1989). Thus, it is not surprising that the initial stages of accretion are quite similar to the preceding case. Runaway growth proceeds as before (Fig. 6). After $10^5$ years the sizes, numbers, and orbital spacings of the largest bodies are similar. The inclinations of the embryos’ orbits are also quite similar; however, their eccentricities are significantly greater for the case with distant perturbations (compare Figs. 5 and 7). This result is due to the fact that distant perturbations act almost entirely to raise eccentricities, while at this stage of accretion, the large bodies are spaced too widely for viscous stirring (or scattering by close encounters) to be effective.

At $3 \times 10^5$ years, the largest embryos typically have $i \sim 10^{-4}$ and $e \sim 10^{-2}$. There are also several times more bodies in the intermediate size range ($\sim 10^{25}$ g) than are seen at the corresponding time without distant perturbations. Their higher eccentricities decrease the rate at which they are accreted by the embryos. There is not much change in the masses and orbits of the largest bodies between $5 \times 10^5$ and $10^6$ years; their orbits appear to be stable on this time scale. Their eccentricities are still increasing, but this may be an artifact of the model for distant perturbations, which gives positive definite stirring (Appendix B3).

D. Combining Orbital Decay and Distant Perturbations

Our next simulation combines all three phenomena: dynamical friction, distant perturbations, and orbital decay due to drag. The evolution of the swarm is qualitatively predictable: runaway growth, with low inclinations and higher eccentricities of the larger embryos. Orbital decay leads to smoothing out of the distribution of the smaller bodies in semimajor axis. Runaway growth produces $\sim 10^{26}$ g bodies, followed by much slower growth (Figs. 8 and 9). The principal difference from the previous case is that the largest bodies continue to merge on the time scale $\sim 10^6$ years until there are only four with masses $> 10^{27}$ g, constituting more than 90% of the total mass of the system. At this time, the amount of mass lost by orbital decay is $\approx 5\%$, with some suggestion of “steps” in the distribution of the small bodies. This is due to shepherding by the largest embryos, which opposes the decay of small bodies outside their orbits. At $10^5$ years, the largest body has a mass $2.2 \times 10^{27}$ g ($0.37 M_\oplus$). The orbital spacings of the four large bodies are about 12 times their mutual Hill radii, and their eccentricities are low enough so that close approaches are impossible. The criterion of Chambers et al. (1996) implies stability for such a system on time scales $> 10^8$ years.
FIG. 4. Distributions of sizes, eccentricities, and inclinations for Case B. Here the stirring model includes dynamical friction and damping of $e$ and $i$ by gas drag (but no secular decay of orbits due to drag). No distant perturbations are computed between bodies in non-crossing orbits. For bodies larger than the median mass, there is a systematic decrease of $e$ and $i$ with increasing mass due to dynamical friction, allowing runaway growth of a small number of large bodies.
With the value of surface density used in the previous tests, this region would contain about $4M_\oplus$, or twice the combined mass of the terrestrial planets. As the previous test show little mass loss (at least in the absence of fragmentation), we decrease the surface density of the swarm to $8.4 \text{ g cm}^{-2}$ at 1 AU, for a total swarm mass of $2M_\oplus$. Thus, we expect slower growth (and smaller planetary embryos) than for the earlier cases. Except for that difference, this simulation resembled Case D, including dynamical friction, distant perturbations, and orbital decay due to gas drag.

With our artificial starting condition of identical sizes and orbital $e$'s and $i$'s (unlikely in the real Solar System), the difference in time scales causes accretion to proceed as a distinct “wave” propagating outward (Fig. 10). In the earliest stage of growth there is a strong tendency toward equipartition of velocities, leading to runaway growth. Bodies of mass $\sim 10^{26} \text{ g}$ accrete on a time scale of a few times $10^5$ orbital periods. After this stage, growth of the embryos slows due to higher velocities and depletion of the smaller bodies. Because long-range perturbations are included, growth at any given heliocentric distance may be inhibited slightly due to stirring by earlier-formed embryos at smaller distances. The innermost zones are not affected in this way, and so enjoy an advantage in addition to their high surface density and short period.

The stochastic nature of our model produces an early runaway by an extra-large embryo at $a \approx 1.35 \text{ AU}$ at $t = 10^5$ years, in advance of the main “wave” of growth. The lack of interference by neighboring embryos allows it to grow more rapidly, while its perturbations on the bodies in neighboring zones appear to inhibit their runaway growth. Despite these favorable circumstances, its growth effectively halts at a few times $10^{26} \text{ g}$, just as it does for the other embryos. Figure 11 shows the evolution of the eccentricities and inclinations versus mass, and the masses versus semimajor axis, for the discrete bodies. After $10^6$ years, there are 21 embryos with mass $> 10^{26} \text{ g}$; the median mass is $\approx 6 \times 10^{26} \text{ g} = 0.1M_\oplus$. Their mean orbital spacing is $\approx 0.09 \text{ AU}$ (a few pairs appear to be in horseshoe orbits).

Figure 12 shows mass versus time for three representative discrete bodies near the inner edge of the swarm, the central region, and outer edge. For the most part, growth is smooth, indicating that most of the mass is accreted from the swarm of small continuum bodies; however, one body shows an abrupt jump in mass between $4 \times 10^5$ and $5 \times 10^5$ years, due to a collision with another discrete body of comparable mass. The pattern of growth is similar for all three bodies, but with progressively later starting times at larger heliocentric distances. An initial stage of rapid runaway growth is followed by a longer period of much slower growth. The decrease in growth rate seems to coincide with the smaller bodies' attaining relatively high eccentricities (\(\geq 0.01\)) that are nearly independent of mass.
FIG. 6. Distributions of sizes, eccentricities, and inclinations for Case C. This case is similar to Case B, but with distant perturbations between bodies in non-crossing orbits included. Runaway growth occurs as before in the early stages, but the large bodies have higher eccentricities.
Using $B = 2\sqrt{3}$ for our value of $\sigma = 8.4 \text{ g cm}^{-2}$, at 1 AU, this “isolation mass” would be $0.05M_\oplus = 3 \times 10^{36} \text{ g}$, and bodies of this mass would be separated by 6.9 Hill radii ($5.5R_\text{H}$, based on the sum of masses of adjacent bodies), or $\approx 0.025 \text{ AU}$. Numerical integrations of orbits of multiple embryos (Kokubo and Ida 1995, Chambers et al. 1996) show that such spacing would be unstable, leading eventually to chaotic crossing of orbits.

The simulation yields both masses and separations approximately three to four times larger than predicted by Eq. (3), due to radial transport of planetesimals by collisional diffusion and gas drag, nonzero eccentricities of embryos and planetesimals, and collisions between embryos (growth is not due solely to sweeping up of small bodies).

Equation (3) predicts $M_t \propto r^3\sigma^{3/2}$; for our initial $\sigma \propto r^{-3/2}$, $M_t \propto r^{3/4}$, and so the final masses should be about twice as large at 1.5 AU as at 0.5 AU. This is not the outcome of the simulation; the largest bodies have comparable masses at all heliocentric distances. After 10$^8$ years there is still some mass available for accretion in the form of continuum bodies and discrete bodies <$10^{26} \text{ g}$, much of it in the outer half of the swarm; however, the 21 largest bodies constitute 88% of the total mass at this point, and the residual material cannot contribute much to their growth. As can be seen from Fig. 11, the largest discrete bodies have masses $p \approx 3 \times 10^{26} \text{ g}$, with eccentricities in the range 0.002–0.02. There is no obvious systematic variation with $a$. Their separations are typically $\approx 10–15R_\text{H}$, implying distances of closest approach $(\Delta a \approx 2ae) \approx (5–10)R_\text{H}$. Spacings of this order appear to be typical at all stages of growth after the emergence of large bodies by runaway growth. These spacings are somewhat larger than the $\approx 5R_\text{H}$ suggested by Ida and Makino (1993). This difference may be due to the fact that their analysis does not include collisions between embryos. Our simulation also covers much greater model time than theirs; the orbits of the embryos should be less stable on longer time scales. The results of Chambers et al. (1996) imply that the orbits of the embryos at the end of our simulation should be unstable on time scales of $10^8 \text{ years}$ (or less; they assumed circular orbits as a starting condition, while ours have significant eccentricities at this point). It is plausible that further evolution of such a system would eventually yield a system of a few planets. The number of large bodies is small enough so that their long-term stability could be investigated by direct numerical integration of their orbits (indeed, with the limitations and approximations in our code, such integration would be more realistic than continuation of this simulation).

V. CONCLUSIONS

We have carried out simulations of planetesimal accretion in a small part of the Solar System—the region of
FIG. 8. Distributions of sizes, eccentricities, and inclinations for Case D. This case is similar to Case C (dynamical friction and distant perturbations), but with secular decay due to gas drag. The small bodies are depleted at the late stage; most are accreted by the largest bodies. Distant perturbations maintain significant eccentricities ($\sim 0.01$) for the largest bodies.
Inclusion of dynamical friction produces a very different outcome, rapid runaway growth of a small number of planetary embryos. This runaway occurs whether stirring is modeled only by “viscous” interactions in close encounters or distant perturbations are included. For our initial conditions, runaway causes bodies $\sim 10^{24}$ g to separate from the continuum size distribution after a few times $10^4$ years. They grow to masses $\sim 10^{27}$ g on a time scale of $\sim 10^5$ years. The difference in the character of accretion, “orderly” versus “runaway,” is robust. In some test simulations we arbitrarily reduced the dynamical friction coefficients ($K_e$, $K_i$; cf Appendix B2). Runaway growth occurred, albeit more slowly, even for coefficients as small as $\frac{1}{3}$ the nominal values.

Distant perturbations between objects in non-crossing orbits are small compared with stirring by crossing orbits (Weidenschilling 1989), but only if there are equivalent populations of crossing and non-crossing bodies. Once planetary embryos become large enough and relatively isolated, distant perturbations dominate their interactions by default. These affect the later “post-runaway” evolution of embryos by raising their eccentricities, while barely affecting their inclinations. The system of large bodies can be more highly flattened than the “equilibrium” value of $i/e \approx 0.6$. The higher $e$’s allow them to collide with the smaller bodies (and occasionally with each other), reducing the tendency toward “dynamical isolation.”

Inclusion of nebular gas, with secular decay of orbits due to drag, affects primarily the smaller bodies and facilitates sweeping up of the remaining objects between the orbits of the embryos. The differences between the final states of the simulations are not great, and may be due in part to the stochasticity of our model. Still, the number of embryos larger than $10^{26}$ g at the “final” stage ($10^6$ years) of the 30-zone simulations is 8 for the simulation without distant perturbations, 6 for distant perturbations without orbital decay, and 4 for distant perturbations with orbital decay. We would expect inclusion of these effects to decrease isolation and allow growth of larger embryos. The larger 100-zone simulation spanning the entire region of terrestrial planets shows a similar result when the different surface density is taken into account. In Case D and the larger simulation of Section IV, the final mass of the largest body is about three times the “isolation mass” predicted by Eq. (3) for the corresponding surface density. The larger value may be attributed to the combined influences of orbital diffusion by collisions and gravitational scattering, secular decay due to gas drag, and eccentricities of the embryos’ orbits. If this result holds to higher surface densities, then $\sigma \approx 30$ g cm$^{-2}$ at 1 AU could produce planets as massive as Earth; however, this value would imply an initial mass of solid matter $\approx 8M_e$ between 0.5 and 1.5 AU, and some
FIG. 10. Distributions of sizes, eccentricities, and inclinations for the simulation with 100 zones spanning the range 0.5 to 1.5 AU. The initial conditions are similar to the test cases (A–D), except for surface density lower by a factor of 2, so that the total mass of the swarm is $2M_\odot$. The time scale for growth is much shorter near the inner edge, but the final sizes of the largest bodies produced by runaway growth are similar in all parts of the swarm.
FIG. 10—Continued
FIG. 11. Parameters of discrete bodies at different stages of evolution of the swarm. Left: eccentricities and inclinations versus mass at $10^4$, $10^5$, $3 \times 10^5$, and $10^6$ years. Right: Masses versus semimajor axes at the same times.
efficient at sweeping up small fragments brought to their orbits by drag; the maximum efficiency was $\approx 0.4$ for an Earth-like embryo, and could be much lower, depending on the drag parameters of the fragment. Future simulations will include fragmentation to determine whether such a scenario of high initial mass and low accretion efficiency can lead to planetary masses and orbital spacings that resemble those in our Solar System. Other potentially important phenomena are not included in this work, e.g., perturbations by Jupiter, migration of embryos due to tidal torques exerted by nebular density waves (Ward and Hahn 1995). They can, however, be treated in principle by our multi-zone code, and will be added to the model in the future.

One should not overinterpret the late-stage results because of the small number of cases and the stochastic nature of these simulations. After a few times $10^5$ years in our models or even direct integration of their orbits. The present code bridges the gap between the early and late phases of planetary formation, and can supply input conditions for such modeling of the final stage.

G. Wetherill has emphasized that simple models may give more insights than complex ones that include “everything but the kitchen sink” (Wetherill and Stewart 1993). His point is well taken, but we feel that caution should be applied more to the interpretation of such models than to their construction. It is necessary to understand individual processes, but many of them acted in concert to form the planets. At some point it is necessary to try to construct a grand synthesis, recognizing that all simulations, whether simple or complex, are imperfect representations of nature’s reality. We do not present ours as a final answer, but as a step in that direction.

APPENDIX A: COLLISION RATES

A1. Crossing and Non-Crossing Orbits

As described in our earlier papers (Spaute et al. 1985, 1991), the rate of collisions between planetesimals, the locations of collisions, and the orbital elements of colliding pairs are chosen stochastically from probability density functions. These functions cover the phase space of possible orbits, with angular variables (nodes, apsides, and mean anomalies) averaged over all values. Representative orbital elements are computed for each colliding pair, and the new orbit of the merged body conserves the vector angular momentum of their center of mass. The original algorithm assumed that the unperturbed orbits overlapped in space; i.e., the aphelion of the inner orbit was greater than the perihelion of the outer. This is a good approximation in the high-velocity regime; however, when eccentricities are sufficiently low, gravitational focusing allows collisions between bodies in non-overlapping orbits. In such cases, the original algorithm did not allow any collisions. We have modified our method to deal with such cases.
First it is necessary to choose which pairs of orbits allow collisions. In the restricted three-body problem, a well-known result is that close approaches are not possible for two bodies initially in circular orbits separated by more than $2\sqrt{3}$ times their mutual Hill radius [cf. Eq. (2)]. We define for bodies with semimajor axes $a_i$, $a_j$, $\Delta a = |a_j - a_i|$, $\hat{a} = (a_j + a_i)/2$, and assume $(m_1 + m_2)/M \ll 1$, $\Delta a/\hat{a} \ll 1$. Not all orbits with $\Delta a < 2\sqrt{3}R_H$ allow collisions. If $\Delta a$ is less than about $1.8R_H$ (and eccentricities are sufficiently small), the bodies will be in horseshoe-type librating orbits (Appendix C, below). Orbits with $\Delta a$ in the range roughly $2.4R_H$ to $2\sqrt{3}R_H$ do not yield collisions (or close approaches) in a single synodic encounter, but only after eccentricities are built up by a series of encounters (Nishida 1983, Greenzweig and Lissauer 1990). For nonzero initial eccentricities the possibility of collision depends on orbital phases as well as separations. From the numerical results of Hasegawa and Nakazawa (1990) and Ipatov (1994), we infer that for moderate eccentricities ($a < \text{less than a few times } R_H$), close approaches are possible when the closest orbital separation (perihelion of outer minus aphelion of inner) is less than about $2.4R_H$. We use this criterion to define pairs of orbits that allow collisions (and also viscous stirring, cf. Appendix B).

The rate of collisions (i.e., the number of collisions in a timestep) is computed using the methodology of Greenberg et al. (1991). To calculate the new orbit that results from such a collision, it is necessary to determine a location at which the collision occurs. For crossing orbits, the location is chosen stochastically within the region of overlapping probability density functions for the orbits of the interacting bodies (or sub-populations within their respective zones), as described in Paper I. For non-crossing orbits, the collision takes place somewhere between the regions occupied by the unperturbed bodies, due to their gravitational focusing. We compute this location by shifting the probability density functions radially to compute this location by shifting the probability density functions radially to

$$\Delta R = \min(\Delta a, 2.4R_H).$$

This is divided between the interacting populations (continuum and/or discrete bodies) as $\Delta R_1 = \Delta Rm_1/(m_1 + m_2)$, $\Delta R_2 = \Delta Rm_2/(m_1 + m_2)$. This choice ensures that the collisions (or close encounters) occur between the limits of the unperturbed orbits and nearer to the orbit of the more massive body. The overlapping probability density functions are used to determine the location of the collision and the relative velocity of the bodies at the time of collision, chosen so that the angular momentum of the center of mass is the same as for the unperturbed orbits.

A2. Collision Cross-Sections and Rates

The rate of collisions between planetesimals depends on their relative velocity and cross-section (actual sizes enhanced by gravitational focusing). The classic two-body collision rate (Safronov 1969) is valid when the individual velocities can be taken as “random,” i.e., when eccentricities and inclinations are sufficiently large. For bodies orbiting a massive primary, the latter’s tidal forces and “Keplerian shear,” the variation of orbital velocities with distance, become dominant at low $e$ and $i$. There have been attempts to quantify these effects (Wetherill and Cox 1985, Ida and Nakazawa 1989, Greenzweig and Lissauer 1990, 1992). Their results were obtained numerically by integration of many different trajectories, and have been presented in the form of total collision rates integrated over all possible colliding orbits in a swarm of uniform surface density. As we need to model collision rates for particular orbits in a swarm that is not necessarily uniform, these formulations are not suitable for our purposes. Instead, we adopt the analytic model of Greenberg et al. (1991). They considered the accretion of small planetesimals by a massive embryo on a circular orbit, and defined three regimes that depended on the $e$’s and $i$’s of the planetesimals scaled to the embryo’s Hill radius. We retain these regimes, but define them for cases when both bodies may have significant $e$’s and $i$’s.

We define the high-velocity Regime A to include all cases where $a(e_1^2 + e_2^2)^{1/2} > 2.5R_H$, where $e_1, e_2$ are the eccentricities of the two colliding bodies. In this regime, the two-body cross section is sufficiently accurate, and the number of collisions in a timestep is computed using the procedure described in Paper I. At lower values of the combined eccentricities there are two more regimes, B and C, defined by the degree of flattening, i.e., the ratio of the gravitational collisional radius to the thickness of the swarm. We determine the appropriate regime as described by Greenberg et al. (1991), except that we use the combined inclination of the two bodies $|\tilde{e} + \tilde{i}|^{1/2}$ to define the thickness.

As described in Paper I, the circumstances of each collision involving continuum bodies are chosen stochastically from populations with distributions of $e$ and $i$ spanning some range about the mean value. The code chooses the $e$ and $i$ values first, then determines the relative velocity, cross sections, the appropriate regime, and collision rate. In general, collision rates increase with decreasing velocities, so rapid accretion at a given size can selectively deplete that population of the members with lower than average $e$ and $i$, in a reverse analog of evaporative cooling.

As the rates are more sensitive to inclination than eccentricity, this effect can produce temporary increases in $i$/e seen during the most rapid phase of runaway growth in some simulations.

We have tested the collision rate versus relative velocity in detail. Figure A1 shows the behavior of the total collision rate for a swarm of small planetesimals encountering a $10^{27}$-g embryo on a circular orbit, as a function of eccentricity of the swarm. The solid curve shows the rate with no dispersion about the mean, while the dashed curve is for a distribution of eccentricities ranging from 0.37 to 1.63 times the mean,

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**FIG. A1.** Total collision rate (per year) on a target embryo as a function of eccentricity of the planetesimal swarm. The embryo is in a circular orbit at 1 AU, and has mass $10^7$ g and density 3.5 g cm$^{-3}$. The planetesimals, of negligible mass, have uniform surface density, with $10^{12}$ bodies per $10^{-1}$-AU interval of semimajor axis at 1 AU. Solid line: uniform eccentricity, with $i = 0.5e$. Dashed line: distribution of eccentricities, uniform between 0.37 and 1.63 times the mean value, with inclinations chosen from a similar distribution with $i = 0.5e$. A, B, and C denote the three velocity regimes defined by Greenberg et al. (1991).
as assumed for continuum bodies in our simulations. The range of $e$ yields a higher collision rate than for the mean value because of the increased contribution of the lower-velocity particles, in agreement with the numerical results of Greenzweig and Lissauer (1992); however, the peak near $e = 10^{-3}$, where the distribution of eccentricities gives a higher collision rate than the limit as $e \to 0$, appears to be an artifact of our algorithm. This effect, amounting to a factor $\approx 2$ over a very limited range, has no significant impact on our results.

For a model with spatial resolution, the distribution of collisions in orbital phase space is as important as their total number. We have compared our collisional algorithm with numerical integrations of trajectories in the three-body problem (K. Ohtsuki, unpublished calculations). Figure A2 shows the results of both methods, with orbits binned in zones of semimajor axis that are narrow compared with the Hill radius. The numerical integrations show more variation with $\Delta a$ due to the complex trajectories that are possible in the three-body problem (Nishida 1983), but the distributions of collision rates with orbital separation are in very good agreement at values of eccentricity such that the particles’ radial excursions are comparable to or smaller than the Hill radius. Note that our algorithm does not allow collisions of bodies with small $\Delta a$, which are in “horseshoe” orbits (see Appendix C).

We also show collision rates for the high-velocity regime; $e = 0.1$, $ae = 18R_H$. The left-hand plot is a numerical evaluation of Greenzweig and Lissauer’s (1990) analytic model; while the right-hand plot shows results of our algorithm. Our model yields somewhat higher mean relative velocities (by about 15%, weighted by collision frequency) and collision rates also about 15% lower than the analytic result. The reason for this difference is not clear. The analytic calculation assumes that $\Delta a/a < 1$, and that the pattern of orbits is symmetrical about the embryo’s orbit. This is a good approximation in the low-velocity regimes, but breaks down at high velocities, which sample a much wider range of phase space. Our numerical calculation shows asymmetry due to the differences in semimajor axes of the orbits in the swarm. The outer bodies have higher radial excursions through a larger volume of space than the inner zones.

APPENDIX B: GRAVITATIONAL STIRRING

Gravitational perturbations change eccentricities and inclinations of bodies in orbits that cross or make close approaches. These changes can be expressed as two phenomena: “viscous stirring” that tends to increase $e$’s and $i$’s, and “dynamical friction” that tends to equalize kinetic energies of random motion among bodies having different masses and semimajor axes that are narrow compared with the Hill radius. The numerical integrations show more variation with $\Delta a$ due to the complex trajectories that are possible in the three-body problem (Nishida 1983), but the distributions of collision rates with orbital separation are in very good agreement at values of eccentricity such that the particles’ radial excursions are comparable to or smaller than the Hill radius. Note that our algorithm does not allow collisions of bodies with small $\Delta a$, which are in “horseshoe” orbits (see Appendix C).

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B1. Viscous Stirring

Stewart separates the radial and transverse components of eccentricity stirring; we simplify the notation by combining these terms. His viscous stirring rates can be expressed as

$$\frac{de^2}{dt} = C(m_1 + m_2)e^2J_r,$$

$$\frac{di^2}{dt} = C(m_1 + m_2)i^2J_r,$$

where the subscript 1 refers to the bodies being stirred and 2 to the population of stirrers. The coefficient $C$ is given by

$$C = \frac{16G^2\rho \ln \Lambda}{VK(e_1^2 + e_2^2)^{3/2}}, \quad (B3)$$

where $G$ is the gravitational constant, $V$ the local Kepler velocity, $\rho_2$ the spatial mass density of stirring bodies, and $\Lambda$ the ratio of maximum encounter distance to the minimum that would result in collision of the bodies (cf. Section II of Stewart and Wetherill 1988 for a more complete discussion).

$J_r$ and $J_i$ are definite integrals that depend on the parameter $\beta$, where

$$\beta^2 = \frac{i^2_1 + i^2_2}{e^2_1 + e^2_2}. \quad (B4)$$

In our notation, $J_r = J_{r2} + 4J_{r1}; J_i = J_{i2} + 2\beta^2$, where the integrals $J_r, J_i, J_e$ are defined in Appendix C of Wetherill and Stewart (1993). They must be evaluated numerically for any given value of $\beta$.

B2. Dynamical Friction

The corresponding expressions for dynamical friction are

$$\frac{de^2}{dt} = C(m_e^2 - m_i^2)Ke,$$

$$\frac{di^2}{dt} = C(m_e^2 - m_i^2)Ke. \quad (B6)$$

Stewart defines another set of definite integrals $K_e, K_r, K_i$ in our notation $K_r = K_e/2 + 2K_i; K_e = K_i/\beta^2$.

Figure B1 shows the $J$ and $K$ coefficients as functions of $\beta$. The curves for $J_r$ and $J_i$ cross at $\beta = 0.6$; when viscous stirring dominates, one finds $i/e \approx 0.6$. The steep rise in $J_e$ at small values of $\beta$ shows the strong tendency for viscous stirring to increase inclinations in a highly flattened system. The rates of change of $e_1^2$ and $i_1^2$ due to viscous stirring and dynamical friction are not directly proportional to the values of the coefficients, since the dynamical friction terms depend on the difference in kinetic energies ($m_e^2, m_i^2$) of the stirrers and stirred. Still, the large value of $K_e$ relative to $J_e$ shows that Stewart’s stirring model (Appendix C of Wetherill and Stewart 1993) gives a strong tendency toward equipartition. This behavior is seen in the simulations that include dynamical friction.

These stirring rates are derived on the assumption that the stirred body’s orbit is surrounded by a continuous swarm of stirrers. The expressions must be modified to account for stirring between zones of finite extent that may not overlap completely. In determining $\rho_2$, the spatial density of stirrers in Eq. (B3), we must define the appropriate volume. Consider two sets of particles in the same zone, i.e., spanning the same range of $a$, but with different $e$’s and $i$’s. If $e_2 > e_1, i_2 > i_1$, then the stirrers occupy a (roughly toroidal) volume that surrounds the stirred population.

The density of stirrers “felt” by the stirred bodies is then straightforward. If $e_2 < e_1, i_2 < i_1$, then the stirrers fill only part of the volume occupied by the stirred bodies, and the mean stirring rate is diminished. By this reasoning, we infer that the appropriate volume to use in computing $\rho_2$ is the larger of the two volumes occupied by stirrers and stirred bodies. If the two populations span different ranges of $a$, then the larger volume may not surround all of the smaller. In that case, the stirring rates are also proportional to the fraction of the smaller volume that is overlapped by the larger. Since the particle populations are assumed to have the same mean orbital planes, we take this factor to be the fraction of the heliocentric range (perihelion to aphelion) in the overlap region.

We apply the expressions for viscous stirring and dynamical friction to orbits that can approach within 2.4 times their mutual Hill radius. This includes a class of orbits that do not cross when unperturbed, but for which their mutual perturbations allow close encounters (or even collisions), and so are not treatable as “distant perturbations” (see below). These orbits do not contribute much to the stirring unless $R_H \approx a(e_1 + e_2)$. In this low-velocity regime, Ida (1990) found that the rates of viscous stirring
FIG. A2. Collision rates versus separation in semimajor axis (normalized to the Hill radius $R_H$). Embryo and swarm have the same parameters as in Fig. A1; eccentricities are monodisperse, with $i = 0.5e$. The embryo is located at $a = 0$. Top row: Results of numerical integrations of orbits (for $e = 0.001$ and $0.01$) and evaluation of Greenzweig and Lissauer (1990) analytical accretion rate (for $e = 0.1$). Bottom row: Results of our code's collisional algorithm for the corresponding values of $e$. In plots of the low-$e$ cases, which are symmetrical about $x = 0$, only outer orbits are shown; for $e = 0.1$, both inner and outer orbits are shown (see text).
and dynamical friction reached limiting values. We apply limits to the stirring rates as described in Appendix D of Wetherill and Stewart (1993). In essence, we do not allow the rates to fall below their values computed for random velocity equal to $2\Omega R_H$.

B3. Distant Perturbations

For orbits that do not allow close encounters, dynamical friction is negligible (Ida 1990), but long-range perturbations allow stirring of velocities. Weidenschilling (1989) suggested that in a continuous swarm of bodies, the stirring by distant perturbations could be accounted for by impulse. We choose the orientation of each encounter randomly, subject to the following constraints: The impact parameter is chosen within a rectangle of width $2D_{\max}$, whichever is smaller. In most cases, the expectation value of the number of such approaches during a timestep is <<1, so we use Poisson statistics to “score” whether such an encounter occurs. The orbital elements of the two bodies and the location of their encounter (defined by their center of mass) define the magnitude and direction of their relative velocity. The two-body scattering formula (e.g., Greenberg et al. 1978) determines the velocity impulse given each body.

The changes in their orbital elements depend on the direction of the impulse. We choose the orientation of each encounter randomly, subject to the following constraints: The impact parameter is chosen within a target area perpendicular to the relative velocity vector of the unperturbed orbits. The central part of the target that would result in collision is excluded. If the vertical range of the bodies ($\Delta y$, where $\Delta y = (y_1 - y_2)^2$) exceeds the maximum encounter distance $r_{\max}$, the target is circular with radius $r_{\max}$. If $\Delta y < r_{\max}$, the target is rectangular, of width $2r_{\max}$ and height $2\Delta y$. This procedure limits the possible out-of-plane deflections when the orbits are nearly coplanar.

At low encounter velocities, the two-body scattering model may not give accurate results for the outcome of any single encounter, but the average outcome of many encounters is in good agreement with detailed orbital integrations (Wetherill and Cox 1984). The two bodies receive impulses in opposite directions, with

$$\Delta V_1 = \left(\frac{m_2}{m_1 + m_2}\right) \Delta V,$$

$$\Delta V_2 = \left(-\frac{m_1}{m_1 + m_2}\right) \Delta V,$$

thereby conserving angular momentum. These encounters do not involve any explicit computation of dynamical friction, which is a phenomenon characteristic of a continuum distribution of orbits, rather than individual bodies; however, the less massive body always experiences the greater deflection in an encounter. In the limit of a large number of encounters,

![FIG. B1. Coefficients for stirring of eccentricity and inclination by viscous stirring ($J_e$, $J_i$), and dynamical friction ($K_e$, $K_i$), adapted from Wetherill and Stewart (1993). $\beta$ is the ratio of inclination to eccentricity. The respective pairs of curves cross near the equilibrium value of $\sin i/e \approx 0.6.$](image-url)
allowed. We have used their results to test our algorithm for gravitational stirring.

The case of a single embryo is shown in Fig. B2. Here and in the following cases, the small bodies that approach the embryo's orbit within 2.4\(R_H\) interact with it by viscous stirring (Appendix B1) and dynamical friction (Appendix B2). More distant bodies are stirred by long-range perturbations (Appendix B3). There is also mutual stirring by the small bodies, but its effect is small compared with that of the massive embryo. The input parameters match those of Ida and Makino (1993); comparison with their Fig. 3b shows good agreement with their stirring rate. Our computed stirring rate matches theirs quite well, but appears to underestimate the size of the region that is strongly stirred by the embryo. Figures B3 and B4 show results for two embryos with initial separations of 10 and 2 Hill radii, which yield good agreement with Fig. 1 of Kokubo and Ida (1995), although treating the small bodies as a continuum neglects a small number of objects that acquire higher eccentricities by close approaches to the embryos. We note that in these examples we allow the small bodies to be effectively stirred by the single embryo or by either of the pair when they formally can encounter only one. If the largest body was assumed to be no more effective at stirring than the second largest, then those small bodies that could encounter only one embryo would hardly be stirred at all. We note that Stewart’s viscous stirring equations appear to give a reasonably good estimate of the rate of stirring by a single embryo, even though such a case does not meet the conditions assumed in the derivation of those equations.

this effect might result in equipartition of energies. No such trend is seen among the discrete bodies in Case A (Fig. 3), however; apparently the size dependence of \(e\) and \(i\) seen in the other cases is due to their interaction with the continuum by dynamical friction. In our simulations there are significant stochastic variations of orbits due to encounters, so that smaller discrete bodies tend to have greater dispersion about the mean values.

B5. Tests of Gravitational Stirring

Ida and Makino (1993) and Kokubo and Ida (1995) studied the stirring of a swarm of small planetesimals by massive embryos. The former considered a simple embryo, while the latter included multiple embryos. Both studies used direct integrations of trajectories, in which mutual perturbations of the small bodies were considered, although collisions were not allowed. We have used their results to test our algorithm for gravitational stirring.

The case of a single embryo is shown in Fig. B2. Here and in the following cases, the small bodies that approach the embryo's orbit within 2.4\(R_H\) interact with it by viscous stirring (Appendix B1) and dynamical friction (Appendix B2). More distant bodies are stirred by long-range perturbations (Appendix B3). There is also mutual stirring by the small bodies, but its effect is small compared with that of the massive embryo. The input parameters match those of Ida and Makino (1993); comparison with their Fig. 3b shows good agreement with their stirring rate. Our computed stirring rate matches theirs quite well, but appears to underesti-
The code does not allow collisions between discrete bodies with semimajor axes that differ by less than this value; nor do discrete bodies accrete continuum bodies from zones that lie wholly within this range of $a$.

Librating bodies are also protected from gravitational stirring. The criterion is slightly more restrictive; from Ida and Nakazawa (1989) and

Finally, we apply our model to a system of many embryos with initial spacing of four times their mutual Hill radii. Kokubo and Ida (1995) found that mutual perturbations caused such a system to become chaotic, with embryos making close approaches and undergoing strong scattering, in fewer than 1000 orbital periods (cf. their Fig. 4b). Our simulations with equivalent parameters typically require between 2000 and 3000 orbital periods to become chaotic. The transition to chaos is stochastic and abrupt, as a single scattering event between two embryos may allow them to approach other neighbors in a "chain reaction." The single example of Kokubo and Ida suggests that our algorithm slightly underestimates the stirring rate, but does not allow a definite conclusion. The final result is the same in any case (Fig. B5).

APPENDIX C: HORSESHOE AND TROJAN ORBITS

Bodies with closely matched semimajor axes may have librating horseshoe- or Trojan-type orbits, and thus be protected from close approaches. The spatial resolution of our code allows us to include such effects. From the results of Ida and Nakazawa (1989) we infer that two bodies will be in libration if

$$\Delta a \leq \left[ \frac{8}{2a(e_1 + e_2)/R_H + 2.5} \right]^{1/2} R_H. \quad (C1)$$

FIG. B4. Stirring by two embryos with initial separation $2R_H$. A close encounter gives the embryos high eccentricities, which are damped by dynamical friction. The end result is an increase in their orbital separation. Compare with Fig. 1 of Kokubo and Ida (1995).

FIG. B5. A system of many embryos with initial separation $4R_H$. Mutual stirring results in close approaches after about 2000 years. Compare with Fig. 4 of Kokubo and Ida (1995). In our simulation, transition to chaotic crossing orbits occurs at $t \sim 2000$ years.
Nishida (1983) we infer that librating orbits can experience substantial deflections that may disrupt the libration unless

\[ \Delta a \leq \frac{8}{ \Delta \theta (e_1 + e_2)/R_{11} + 3.5 } R_{11}. \]  
\[ (C2) \]

Librations stabilize pairs of bodies, yet they may be destabilizing in a many-body swarm. Each synodic encounter changes the semimajor axes of the pair, with the inner becoming the outer, and vice versa. For librating pairs of discrete bodies, we take the synodic period to be

\[ P_{\text{syn}} = \frac{2 \pi a^2}{3 \nu a} \]  
\[ (C3) \]

If a timestep exceeds \( P_{\text{syn}} \), their semimajor axes are changed by the amounts

\[ \Delta a_i = \text{sign}(a_2 - a_1) \frac{m_i}{m_1 + m_2} \Delta a, \]  
\[ \Delta a_2 = - \frac{m_1}{m_2} \Delta a_1. \]  
\[ (C4) \]

If \( \Delta t < P_{\text{syn}} \), they are changed if a random number is chosen \( \leq \Delta t / P_{\text{syn}} \). At some stages of our simulations, when discrete bodies are numerous, a significant fraction \((\approx 1/2)\) may be in librating orbits at any given time. These pairings tend to be short-lived, as \( \Delta a \) may be changed by collisions or gravitational encounters with other bodies, or increasing \( e \)'s will shrink the protected range of \( \Delta a \). As most librating pairs are protected from both collisions and mutual stirring, the net effect is a modest increase in the time scale of growth of the larger bodies, with little or no qualitative change in the final outcome. Only in one case did we find a libration that appeared to be permanent: two \(-10^2\) g bodies that maintained nearly identical orbits for \( >5 \times 10^3 \) years. We note that the criterion of Eq. (C1) has been tested only for short-term stability; the actual conditions for long-term stability of librations may be more restrictive.

**APPENDIX D: DISTANT PERTURBATIONS AND "SHEPHERDING"**

Distant perturbations increase \( e \)'s (and, to a much lesser degree, \( i \)'s). Conservation of the Jacobi parameter implies that this stirring is accompanied by changes in \( a \) as well (it is not necessary to assume that one body is massless). The changes in \( a \) act to increase the separation of the orbits. If the initial separation is \( \Delta a_0 \), and a synodic encounter results in changes in \( e \) and \( i \), the final separation is (Hasegawa and Nakazawa 1990)

\[ \Delta a = \left[ \Delta a_0 + \frac{4 a^2}{3} \Delta (e^2 + i^2) \right]^{1/2}. \]  
\[ (D1) \]

In general, \( \Delta a < a \Delta e \), so the distance of closest approach tends to decrease, even though the mean separation increases; however, damping of eccentricities by other means, e.g., gas drag, collisions, or dynamical friction, can produce a net “shepherding” effect that makes the bodies effectively repel each other.

In our code, stirring of \( e \) and \( i \) by distant perturbations (Appendix B3) is accompanied by changes in \( a \). For pairs of discrete bodies, the various quantities in Eq. (D1) are straightforward. For stirring of the continuum by discrete bodies, we evaluate the change in \( a \) using the value of \( \Delta a \) at the edge of the zone farthest from the stirring body, as this is the value that may shift bodies from one zone to another. The fraction of bodies shifted in a timestep is \( \Delta a - \Delta a_0 \) divided by the zone width. We note that viscous stirring must also result in changes in \( a \) that can be regarded as radial diffusion in the swarm. We do not model this effect; when viscous stirring is dominant, the numbers of bodies in continuum size bins of adjacent zones are similar, so the net diffusive flux between zones would be small. Viscous diffusion could cause significant mass transport only near gaps in the swarm or steep gradients in the population of planetesimals. We find that when such gaps or gradients form, viscous stirring of the continuum is generally less important than distant perturbations. Also, by that stage most of the swarm mass is in the discrete bodies. Their mutual encounters, both close and distant, include the appropriate changes in \( a \).

**APPENDIX E: GAS DRAG AND BOUNDARY CONDITIONS**

The gaseous solar nebula (when present) has the effect of damping eccentricities and inclinations. Also, the non-Keplerian rotation of the pressure-supported gas causes secular decay of semimajor axes. We employ the expressions of Adachi et al. (1976; their Eqs. 2, 5) for the rates of change of these elements (with the correction for \( \text{de} / \text{dt} \); Eq. 11 of Kary et al. 1993). We assign each heliocentric zone its own gas density according to a power law based on the surface density of the nebula (the initial surface density of planetesimals follows the same power law). The gas density at 1 AU is taken to be \( 1.18 \times 10^{-9} \) g cm\(^{-3} \), with deviation from Keplerian rotation of \( 2 \times 10^{-3} \), the same values used by Wetherill and Stewart (1993).

Simulations were performed with two models. The first set had damping of \( e \) and \( i \), but no changes in \( a \) due to drag. The second set included orbital decay. For those simulations with orbital decay, the radial velocity gives a decrease in semimajor axis for any body during a timestep. For discrete bodies, this \( \Delta a \) (drag) is added to the other changes due to collisions, distant perturbations, etc. Continuum bodies may be shifted into the next zone inward. The fraction of bodies in any size bin and heliocentric zone that is shifted is \( \Delta a \) (drag) divided by the zone width. If more than one body per timestep is shifted between a given pair of zones during a timestep, the computed number (including fractional bodies) is used. If, as is usually the case, the number of bodies shifted per timestep is less than one, we choose a random number to decide whether one body or none is shifted.

Orbital decay requires careful choice of boundary conditions. Bodies that cross the inner edge of the swarm due to gas drag are removed (the reflective boundary refers only to collisional and gravitational scattering). The outer part of the swarm would become depleted of small bodies if they are not replenished from regions at larger heliocentric distances (a 10 km sized body would move more than 0.01 AU in 10\(^3 \) years). We assume that the region of simulation is part of a wider swarm, and so replenish the outermost zone. The gradient in gas density implies that the amount of mass entering a zone is slightly less than the amount leaving it (\( =98\% \) for our nominal model). To prevent numerical instabilities due to local fluctuations in the planetesimal population, we scale the rate of replenishment at any given size to the mass transfer rate averaged over the outermost ten zones. This replenishment is done for continuum size bins only, not for discrete bodies.

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