A HIGH-ORDER ANALYTICAL MODEL FOR THE SECULAR DYNAMICS OF IRREGULAR SATELLITES

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ABSTRACT

We develop an analytical model for the long-term (secular) dynamics of irregular satellites of the giant planets. The disturbing potential in this model is represented by a high order (in semimajor axis, eccentricity and inclination) Legendre expansion. We use a third-order Hori's averaging method to eliminate terms in the original equations that are irrelevant for the long-term dynamics and to construct new second and third-order secular terms. The resulting secular equations are valid for both direct and retrograde orbits (of any inclinations) and for eccentricities up to ≈ 0.7.

In the present paper we describe the mathematical background of our method and test it in several applications. The method uses a Hamiltonian formulation of dynamics. The original Hamiltonian and its high-order secular forms are represented by series that have self-similar functional forms. The coefficients of these series are calculated by using an algebraic manipulator. This approach allows us to iterate the Hori's perturbation method to high orders.

To test our method, we *(i)* calculate the precession frequencies of orbits of the irregular satellites at Jupiter, and *(ii)* determine the the dynamical structure of the Kozai resonance. We show that this resonance occurs at progressively larger (proper) inclinations with increasing separation of the satellite from the parent planet. These results are compared to those obtained by numerically integrating the exact equations of motion. Our theory will be particularly useful for determining the locations and strengths of secular resonances in the space occupied by distant satellite orbits. Several irregular satellites have been trapped in secular resonances by some, likely primordial mechanism.

1. Introduction

Following Burns (1986), we define the irregular satellites of the outer planets as those moons that are sufficiently far from the planet such that the precession of the orbital plane is dominated by the solar perturbation. These bodies are thought to have been captured by the planet during the last stages of planetary formation, but the exact mechanism of capture is not completely understood. Dynamically, they are characterized by large planetocentric semimajor axis and high eccentricities. Some irregular satellites have direct orbits (i.e. same direction as the planet orbits the Sun), but many move in retrograde planetocentric orbits. In both cases the inclination can also take very high values; up to 40 degrees for direct orbits, and 140 degrees for retrograde bodies. Carruba et al. (2002) and Nesvorný et al. (2003) have recently shown that the orbits with intermediate inclinations ($\approx$60°-120°) are unstable due to the effects of the Kozai resonance (Kozai 1962).

Notwithstanding their large semimajor axes and high eccentricities the irregular satellites have stable orbits. If chaos exists, it is weak, as in the case of the Jovian satellite Sinope (Saha and Tremaine 1993). This allows researchers to apply perturbation theories and obtain useful approximations of the long-term dynamics of irregular satellites via analytic calculations.

Sophisticated analytical theories for satellite motion date from the 19th century, developed mainly to explain the motion of the Moon (e.g. Delaunay 1860, 1867). For the irregular satellites, however, analytical perturbation methods are more challenging. First, the orbital characteristics require that the expansion of the disturbing potential is done to high orders in orbital elements. Second, the complex interaction between the different degrees of freedom of the dynamical system would require a high-order perturbation method and the use of algebraic manipulators.

These difficulties have taken their toll: the long-term dynamics of irregular satellites is only partially understood. When analytical or semi-analytical models are constructed (e.g. Hénon 1970, Kinoshita and Nakai 1991, Carruba et al. 2002, Yokoyama et al. 2003), their results are more qualitative than quantitative. These models have mainly been employed to help interpret numerical results, and not to make useful predictions about the dynamical behavior of the system. Recently, Cuk and Burns (2004) constructed an empirical analytical theory, in which ad-hoc high-order perturbative terms are added to the variational equations. Comparisons with numerical simulations showed that the model, although empirical, was very precise.

In the present paper we develop a new, high-order analytical model for the secular (i.e. long-term) dynamics of irregular satellites of the outer planets. Although the aims are similar to the work of Cuk and Burns (2004), the present model is built in a self-consistent form. In Section 2, we present Kaula's expansion of the disturbing function for direct and retrograde orbits. In Section 3, we construct the Hamiltonian system. Our perturbation theory is described in Section 4. The averaging over the mean anomalies and construction of the secular dynamical system is described in Section 5. Two applications of our secular equations are discussed. In Section 6, we use the equations to study the Kozai resonance (Kozai 1962). In Section 7, we calculate the precession frequencies of orbits of the irregular satellites at giant planets. Conclusions are given in Section 8.

2. Kaula's Expansion of the Disturbing Function

We consider the restricted three-body problem comprised of a small satellite orbiting a planet of mass m_0 and perturbed by the Sun with mass m_1. Let $\vec{r}$ be the instantaneous planetocentric position vector of the satellite and $\vec{r_1}$ that of the Sun. The disturbing potential function of the satellite's planetocentric Keplerian orbit is defined as:

$$R = Gm_1\left(\frac{1}{|\vec{r}-\vec{r_1}|} - \frac{\vec{r}\cdot\vec{r_1}}{r_1^3}\right), \tag{1}$$

where r and r_1 are the moduli of $\vec{r}$ and $\vec{r_1}$, respectively, and G is the gravitational constant. When the ratio r/r_1 is much smaller than unity (such as in the case of a planetary satellite), it is useful to expand R in Legendre polynomials:

$$R = \frac{Gm_1}{r_1}\sum_{l=2}^{\infty}\left(\frac{r}{r_1}\right)^l P_l(\cos\theta), \tag{2}$$

where θ is the angle between $\vec{r}$ and $\vec{r_1}$. This expansion can be expressed in terms of the orbital elements of a satellite (Kaula 1961, 1962). For both prograde and retrograde satellite orbits, this is given by:

$$\begin{aligned} R &= \frac{Gm_1}{a_1}\sum_{l=2}^{\infty}\alpha^l\sum_{n=0}^{l}(-1)^{l-n}\kappa_n\frac{(l-n)!}{(l+n)!} \\ &\times \sum_{p,p_1=0}^{l}F_{l,n,p}(I)F_{l,n,p_1}(I_1)\sum_{q,q_1=-\infty}^{\infty}X_{l-2p+q}^{l,l-2p}(e)X_{l-2p_1+q_1}^{-l-1,l-2p_1}(e_1)\quad\cos\theta, \end{aligned} \tag{3}$$

where

$$\theta = (l-2p_1+q_1)M_1 - (l-2p+q)M + (l-2p_1)\omega_1 - (l-2p)\omega + n(\Omega_1-\Omega), \tag{4}$$

and $\alpha = a/a_1$. The κ_n function appears as a consequence of the passage from exponential to trigonometric functions, and it is equal to unity for $n = 0$ and equal to 2 for all other values of the index. The satellite's orbital elements are given by its semimajor axis a, eccentricity e, inclination I with respect to an invariant reference system (e.g., the Laplacian plane), mean anomaly M, argument of the pericenter ω, and longitude of the ascending node Ω. We use the subscript 1 to denote the orbital elements of the Sun in the planetocentric reference frame.

The functions $F_{l,n,p}(I)$ are complicated functions of the inclinations. According Murray and Dermott (1999), these functions are:

$$F_{l,n,p}(I) = \frac{(\sqrt{-1})^{l-n}(l+n)!}{2^l p!(l-p)!}\sum_k(-1)^k\begin{pmatrix}2l-2p\\k\end{pmatrix}\begin{pmatrix}2p\\l-n-k\end{pmatrix}\xi^{3l-n-2p-2k}\eta^{n-l+2p+2k}, \tag{5}$$

where the sum is limited to $k \in [\max(0, l-n-2p), \min(l-n, 2l-2p)]$. The inclinations appear in $\xi = \cos\frac{1}{2}I$ and $\eta = \sin\frac{1}{2}I$.

The dependence on the eccentricities is given via the Hansen coefficients $X_c^{a,b}$, which can be defined as:

$$X_c^{a,b}(e) = e^{|c-b|}\sum_{s=0}^{\infty}Y_{s+t,s+u}^{a,b}e^{2s}, \tag{6}$$

where $t = \max(0, c-b)$, $u = \max(0, b-c)$, and $Y^{a,b}_{s+t,s+u}$ are the Newcomb operators which can be determined via simple recurrence relations (see Hughes 1981, Dermott and Murray 1999).

2.1. Expansion in Complex Variables

It will prove useful to translate the cosines in Eq. (3) to their exponential counterparts and to work with complex coefficients. We begin introducing new indexes defined by:

$$\begin{array}{ll} i_1 = l - 2p_1 + q_1 & \in (-\infty, \infty) \\ i_2 = l - 2p + q & \in (-\infty, \infty) \\ i_3 = l - 2p & \in [-l, l] \\ i_5 = l - 2p_1 & \in [-l, l] \end{array} \tag{7}$$

where the last two indexes only take odd (even) values if l is odd (even). We also group the dependence on the inclination and eccentricity of the Sun inside the coefficients. This is useful to obtain a compressed version of the expansion, and introduces no approximations since the planetary orbit will be assumed constant. With this in mind, we can write

$$R = Gm_1 \sum_{l=2}^{\infty} a^l \sum_{n=0}^{l} \sum_{i_1,i_2=-\infty}^{\infty} \sum_{i_3,i_5=-l}^{l} A_{l,n,i_5}(a_1, I_1, e_1) \;\; F_{l,n,i_3}(I) \;\; X^{l,i_3}_{i_2}(e) \;\; \cos\theta, \tag{8}$$

where

$$\theta = i_1 M_1 - i_2 M + i_5 \omega_1 - i_3 \omega + n(\Omega_1 - \Omega). \tag{9}$$

We now pass from the $\cos\theta$ to exponentials. The main advantage of this transformation is that it allows us to group inside the new coefficients the dependence with ω_1, Ω_1. After a few algebraic calculations, we obtain:

$$R = Gm_1 \sum_{l=2}^{\infty} a^l \sum_{i_1,i_2=-\infty}^{\infty} \sum_{i_3,i_4=-l}^{l} B_{l,i_4}(a_1, I_1, e_1, \omega_1, \Omega_1) \;\; F_{l,-|i_4|,i_3}(I) \;\; X^{l,-i_3}_{-i_2}(e) \;\; E^{\sqrt{-1}\psi}, \tag{10}$$

where B_{l,i_4} are the new complex coefficients and we have used the notation $E^x \equiv \exp x$. The argument of the periodic terms now read

$$\psi = i_1 M_1 + i_2 M + i_3 \omega + i_4 \Omega. \tag{11}$$

2.2. Separating Prograde and Retrograde Satellite Orbits

Functions $F_{l,n,p}(I)$ in equation (5) are valid for any value of I. However, we note that for low-inclination prograde orbits (i.e. $I \approx 0$), we have that $\eta \ll 1$ and $\xi \approx 1$. Conversely, for low-inclination retrograde orbits (i.e. $I \approx \pi$), we have that $\xi \ll 1$ and $\eta \approx 1$. This property allows us to expand $F_{l,n,p}(I)$ as

power series of ξ or η depending on whether the satellite is retrograde or prograde. For example, in the case of retrograde orbits, and for any non-negative integers j, k, we can write:

$$\xi^j \eta^k = \xi^j (1-\xi^2)^{k/2} = \sum_{i=0}^{\infty} C_i^{(k)} \xi^{2i+j} \tag{12}$$

where the coefficients have the following recurrence relation:

$$C_0^{(k)} = 1 \quad ; \quad C_i^{(k)} = -\frac{(k-i+1)}{k} C_{i-1}^{(k)}. \tag{13}$$

This series is valid in the interval $I \in [\pi/2, \pi]$. For direct orbits, the expressions above acquire the form:

$$\xi^j \eta^k = \eta^k (1-\eta^2)^{j/2} = \sum_{i=0}^{\infty} C_i^{(j)} \eta^{2i+k} \tag{14}$$

with the same coefficients $C_i^{(j)}$ given by (13). This series is valid for $I \in [0, \pi/2]$. In the case of vertical orbits (i.e. $I = \pi/2$) both expressions can be used.

Introducing (12) into (5) we can now determine new constant coefficients $g_{l,n,p,k}$ and write for retrograde orbits:

$$F_{l,n,p}(I) = \sum_{k=0}^{\infty} g_{\,l,n,p,k}\, \xi^k. \tag{15}$$

The same procedure can also be undertaken for direct orbits, yielding a power series in η.

Finally, since most irregular satellites have eccentricities below the limit $e_c \simeq 0.6667$, we can use the expansion of Hansen coefficients in power series of the eccentricities. Thus, we can write:

$$X_c^{a,b}(e) = \sum_{j=0}^{\infty} x_{a,b,c,j}\; e^j, \tag{16}$$

where the expressions for $x_{a,b,c,j}$ can be easily obtained from the Newcomb operators. Note that all values are zero for $j < |c-b|$; however, we will keep the general form for simplicity.

Introducing all these changes into the expression for R, for retrograde orbits we obtain:

$$R = Gm_1 \sum_{l=2}^{\infty} \sum_{j,k=0}^{\infty} \sum_{i_1,i_2=-\infty}^{\infty} \sum_{i_3,i_4=-l}^{l} R_{l,j,k,i_1,i_2,i_3,i_4}\;\; a^l e^j \xi^k \;\; E^{\sqrt{-1}(i_1 M_1 + i_2 M + i_3 \omega + i_4 \Omega)}. \tag{17}$$

where $R_{l,j,k,i_1,i_2,i_3,i_4}$ are the new complex coefficients. In the case of direct orbits, the power series in ξ is replaced by one in η.

As a final comment, it is important to keep in mind certain properties of the Legendre expansion of the disturbing function. Compared to the Laplace version, Kaula's series has the advantage of being convergent for eccentricities below $e_c \simeq 0.6667$ (see Wintner 1941) and all values of the inclinations. The value of e_c

is actually a function of α (see Ferraz-Mello 1994), but it is practically unchanged for all $\alpha < 0.1$. If Bessel functions are used instead of Newcomb operators, the resulting expressions could be used even in the case of eccentricities close to unity. However, the Legendre expansion has two drawbacks. On one hand, its rate of convergence is very slow as function of α; thus it is usually applied only to those cases where the ratio of semimajor axes is very small. Secondly, the literal expression of R is very complex and only a few terms are usually included in models.

3. Hamiltonian Formulation

The Hamiltonian function for the dynamics of a satellite, in the extended phase-space, can be written as:

$$F = -\frac{\mu^2}{2L^2} + n_1\Lambda - R(L, G, H, M, \omega, \Omega) \tag{18}$$

where $\mu = \mathcal{G}m_0$ and $\mathcal{G}$ is the gravitational constant . The mean-motion of the planet is denoted by n_1, and Λ represents the canonical conjugate of time. The Delaunay canonical variables are written in terms of the orbital elements of the satellite as:

$$\begin{aligned} L &= \sqrt{\mu a} \\ G &= L\sqrt{1-e^2} \\ H &= G\cos(I) = G(2\xi^2 - 1) = G(1 - 2\eta^2). \end{aligned} \tag{19}$$

However, we will not explicitly introduce the transformation $(e, \xi) \to (G, H)$ into (17) since it would lead to unnecessary complications. We will then simply write the Hamiltonian as:

$$F = n_1\Lambda + \sum_{i,j,k} \sum_{l_1,\ldots,l_4} f_{i,j,k,l_1,\ldots,l_4} \;\; L^i e^j \xi^k \;\; E^{\sqrt{-1}(l_1 M_1 + l_2 M + l_3\omega + l_4\Omega)}. \tag{20}$$

Note that the first term in (18) has been included into the sum. This is an example of what is sometimes referred to as a *Poisson series* (e.g. Henrard 1989), and it is characterized by a power series in the subset (L, e, ξ) and a Fourier series in the angles.

3.1. Operations on the Hamiltonian Expansion

In order to use Eq. (20) in a perturbation theory and construct a secular model for the evolution of the irregular satellites, we must be able to perform operations on the series, such as differentiation with respect to the canonical variables. The angles pose no problem at all. For example, the derivative of F respect to M will yield a series of the same type as (20) with new coefficients:

$$f^{(M)}_{i,j,k,l_1,\ldots,l_4} = \sqrt{-1}\; l_1 \; f_{i,j,k,l_1,\ldots,l_4}, \tag{21}$$

and similar expressions are also found for the remaining angles.

Derivatives with respect to the canonical momenta are more difficult, and constitute a common problem in this type of analytical work. While the perturbation equations are written in terms of partial derivatives with respect to the canonical momenta L, G, H, our Hamiltonian (20) is written in L, e, ξ. If the expression of R is not restricted to a small number of terms, the transformation $R(L, e, \xi) \rightarrow R(L, G, H)$ would yield very complicated expressions and we lose the original Poisson form. Any posterior differentiation or integration will complicate things even further, up to a point where we would necessarily have to adopt approximations which would limit the validity of the model.

Using Eq. (20) we can write for retrograde orbits:

$$\begin{aligned}
\frac{\partial(L^i e^j \xi^k)}{\partial L} &= (i-j)\, L^{i-1} e^j \xi^k + j\, L^{i-1} e^{j-2} \xi^k \\
\frac{\partial(L^i e^j \xi^k)}{\partial G} &= -\sum_{n=0}^{\infty} j\, C_n^{(1/2)} L^{i-1} e^{j+2n-2} \xi^k - \sum_{n=0}^{\infty} \frac{k}{2}\, C_n^{(-1/2)} L^{i-1} e^{j+2n} \xi^k \\
&\quad + \sum_{n=0}^{\infty} \frac{k}{4}\, C_n^{(-1/2)} L^{i-1} e^{j+2n} \xi^{k-2} \\
\frac{\partial(L^i e^j \xi^k)}{\partial H} &= \sum_{n=0}^{\infty} \frac{k}{4}\, C_n^{(-1/2)} L^{i-1} e^{j+2n} \xi^{k-2}
\end{aligned} \tag{22}$$

where $C_n^{(1/2)}$ and $C_n^{(-1/2)}$ are given by equations (13). For direct orbits, the derivatives are given by the following expressions:

$$\begin{aligned}
\frac{\partial(L^i e^j \eta^k)}{\partial L} &= (i-j)\, L^{i-1} e^j \eta^k + j\, L^{i-1} e^{j-2} \eta^k \\
\frac{\partial(L^i e^j \eta^k)}{\partial G} &= -\sum_{n=0}^{\infty} j\, C_n^{(1/2)} a^i e^{j+2n-2} \eta^k - \sum_{n=0}^{\infty} \frac{k}{2}\, C_n^{(-1/2)} L^{i-1} e^{j+2n} \eta^k + \sum_{n=0}^{\infty} \frac{k}{4}\, C_n^{(-1/2)} L^{i-1} e^{j+2n} \eta^{k-2} \\
\frac{\partial(L^i e^j \eta^k)}{\partial H} &= -\sum_{n=0}^{\infty} \frac{k}{4}\, C_n^{(-1/2)} L^{i-1} e^{j+2n} \eta^{k-2}.
\end{aligned} \tag{23}$$

Notice that these expressions are singular for circular and/or planar orbits. Introducing these derivatives into our Hamiltonian, for retrograde orbits we obtain:

$$\begin{aligned}
\frac{\partial F}{\partial L} &= \sum_{i,j,k} \sum_{l_1,\ldots,l_4} f^{(L)}_{i,j,k,l_1,\ldots,l_4}\ L^l e^j \xi^k\ E^{\sqrt{-1}(l_1 M_1 + l_2 M + l_3 \omega + l_4 \Omega)} \\
\frac{\partial F}{\partial G} &= \sum_{i,j,k} \sum_{l_1,\ldots,l_4} f^{(G)}_{i,j,k,l_1,\ldots,l_4}\ L^l e^j \xi^k\ E^{\sqrt{-1}(l_1 M_1 + l_2 M + l_3 \omega + l_4 \Omega)} \\
\frac{\partial F}{\partial H} &= \sum_{i,j,k} \sum_{l_1,\ldots,l_4} f^{(H)}_{i,j,k,l_1,\ldots,l_4}\ L^l e^j \xi^k\ E^{\sqrt{-1}(l_1 M_1 + l_2 M + l_3 \omega + l_4 \Omega)}
\end{aligned} \tag{24}$$

where the new coefficients are given by

$$f^{(L)}_{i,j,k,l_1,\ldots,l_4} = (i-j+1)\, f_{i+1,j,k,l_1,\ldots,l_4} + (j+2)\, f_{i+1,j+2,k,l_1,\ldots,l_4}$$

$$
\begin{aligned}
f^{(G)}_{i,j,k,l_1,\ldots,l_4} &= -\sum_{n=0}^{n_{max}+1} (j-2n+2)\, C_n^{(1/2)} f_{i+1,j-2n+2,k,l_1,\ldots,l_4} - \sum_{n=0}^{n_{max}} \frac{k}{2}\, C_n^{(-1/2)} f_{i+1,j-2n,k,l_1,\ldots,l_4} \\
&\quad + \sum_{n=0}^{n_{max}} \frac{(k+2)}{4}\, C_n^{(-1/2)} f_{i+1,j-2n,k+2,l_1,\ldots,l_4} \qquad (25) \\
f^{(H)}_{i,j,k,l_1,\ldots,l_4} &= \sum_{n=0}^{n_{max}} \frac{(k+2)}{4}\, C_n^{(-1/2)} f_{i+1,j-2n,k+2,l_1,\ldots,l_4}
\end{aligned}
$$

with $n_{max} = \mathrm{int}(j/2)$. The expressions for direct orbits are analogous. The only difference is a change of sign for $f^{(H)}_{i,j,k,l_1,\ldots,l_4}$.

The main importance of this procedure is that it allows us to find a *self-similar* expression for all the operations involved in the perturbation equations. In other words, the original Hamiltonian and its derivatives, integrals, products, etc., have the same functional form. The corresponding series only differ in values of the the coefficients, which are not dependent on the orbital elements of the satellite. The advantage of this method is that it allows us to iterate the Hori's perturbation theory to a high order.

3.2. Fundamental Frequencies

As a first application of method, we determine the fundamental frequencies in each canonical angle. We restrict F to its truncated form F', defined as all those terms that do not depend explicitly on the set (M, ω, Ω). From Hamilton's equations we can write:

$$
\begin{aligned}
\nu_L \equiv \dot{M} = \frac{\partial F'}{\partial L} \qquad & \qquad \nu_\Lambda \equiv \dot{M}_1 = \frac{\partial F'}{\partial \Lambda} \qquad (26) \\
\nu_G \equiv \dot{\omega} = \frac{\partial F'}{\partial G} \qquad & \qquad \nu_H \equiv \dot{\Omega} = \frac{\partial F'}{\partial H}
\end{aligned}
$$

where F' includes only those terms in F that satisfy $l_1 = l_2 = l_3 = l_4 = 0$. It can easily be seen that $\nu_\Lambda = n_1$ is simply the mean-motion of the planet, ν_L is the orbital frequency of the satellite, and ν_G, ν_H are the frequencies of variation of the argument of pericenter and ascending node, respectively. For example, ν_G is given by:

$$
\nu_G = \sum_{i,j,k} f^{(G)}_{i,j,k,0,0,0,0}\; L^i e^j \xi^k \qquad (27)
$$

and similar expressions hold to the other frequencies.

The continuous curves in Figure 1 represent the values the frequencies, as function of the semimajor axis, for Jovian retrograde satellites with $e = 0.3$ and $I = 170$ deg. Although ν_L and ν_Λ are much larger than ν_G and ν_H for small values of the semimajor axis, for distant moons all these frequencies are roughly of the same order. In particular, for $a = 0.22$ AU (corresponding to $\approx 0.7 R_{Hill}$), the orbital frequency of the satellite is only about 3 times larger than ν_Λ, and this last value is less than twice ν_G and only about four times ν_H. Thus, contrary to what is found other non-resonant three-body systems (for example,

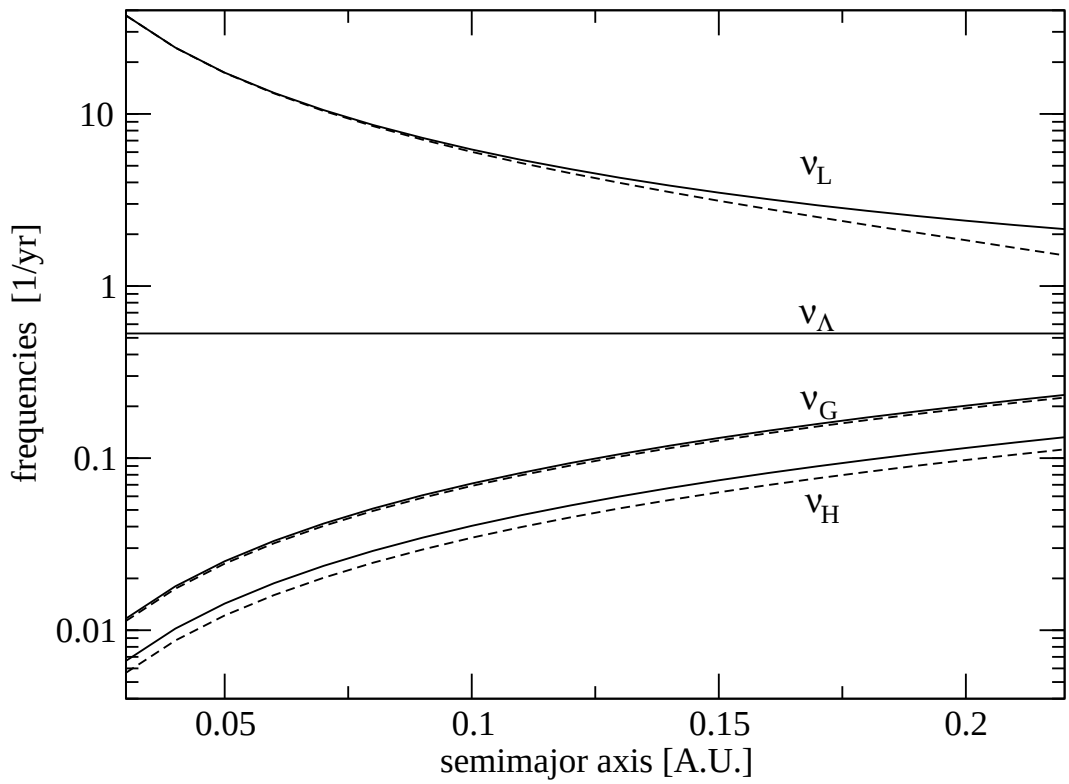

Fig. 1.— Fundamental frequencies of each canonical angle as a function of semimajor axis of the satellite that has $e = 0.3$ and $I = 170$ deg. The satellite orbits a Jupiter-like planet with mass $m_0 = 10^{-3} m_1$ that was placed on the current osculating orbit of Jupiter. Continuous curves represent the values obtained with the complete truncated secular Hamiltonian F'. Broken curves show results determined with our Kernel F_0 (see equations (29)-(30)).

in the asteroid belt), for the irregular satellites we find a significant region of the phase space where all the fundamental frequencies are roughly of the same order, and the separation between "short-period" and "long-period" degrees of freedom becomes blurred. As a consequence, seemingly unimportant terms such as the evection (Delaunay, 1860, 1867) have important contributions in the dynamics of distant planetary satellites.

4. Perturbation Theory

To study the secular dynamics of the irregular satellites, it is usual practice to eliminate all terms which depend explicitly on both mean anomalies. This is performed with an averaging process obtained from the application of a chosen perturbation theory. Analytical works (such as Saha and Tremaine 1993) usually adopt a first-order averaging in the mean anomalies, which simply corresponds a straightforward elimination of all periodic terms in M, M_1. However, it is well known that some of the periodic terms eliminated by

this approximation (such as, e.g., the evection term) have significant effects on the long-term evolution of distant satellites. In Lunar theories, the importance of such long-period terms has been recognized since the times of Delaunay (see also Touma and Wisdom 1998), and shows that it is necessary to go beyond a first-order averaging to have an adequate representation of the secular dynamics. In recent years, a number of second-order secular theories have appeared in the literature, for both the restricted and planetary three-body problems (e.g. Milani and Knežević 1994, Lee and Peale 2003, Cuk and Burns 2004).

Most of these second-order models were based on the canonical theories of Von Zeipel (1916) or Hori (1966), although sometimes a simpler variation of constants was adopted (Cuk and Burns 2004). In all cases, however, the application of the perturbation theory followed three main steps: *(i)* The ratio of semimajor axes α or, alternatively, the ratio of mean-motions n_1/n was usually chosen as the "small parameter" of the perturbation, around which the theory was constructed. The zero-order term corresponds to the kernel. *(ii)* The two-body contribution (including the Λ term) was chosen as the kernel (e.g. unperturbed Hamiltonian function F_0). The unperturbed system was thus degenerate and the fundamental frequencies ν_G and ν_H are zero. An expansion of the disturbing function up to order α^2 yielded what is called the quadrupole approximation, while terms up to α^3 give the so-called octupole approximation. The evection term appears in α^3-terms. *(iii)* As an initial step in the model, a first order averaging was initially performed on the mean anomaly of the satellite (i.e. elimination of M). The second-order theory was only constructed for the remaining three-degree of freedom system.

The choice of $F_0 = -\mu^2/2L^2 + n_1\Lambda$ as the kernel has historical reasons and is a very good approximation for the secular dynamics of main belt asteroids (e.g. Milani and Knežević 1994), Lunar theories (e.g. Delaunay 18860, 1861, Touma and Wisdom 1998), or regular satellites of the outer planets. In the example of the asteroids, ν_G and ν_H are typically several orders of magnitude smaller than the orbital frequencies of the bodies. Thus considering these quantities equal to zero is well justified as a first approximation. However, for irregular satellites of the Jovian planets, we have seen in Figure 1 that all frequencies may acquire values of similar magnitudes. In such a case, the choice of the two-body Hamiltonian as the kernel is no longer a good option. Similarly, first-order averaging over M is also not a good approximation. Once again, this assumes very different timescales for the variation of the different degrees of freedom, a characteristic which is not necessarily correct for irregular satellites.

We thus need to define a new unperturbed Hamiltonian and develop a new perturbation approach. To define the new unperturbed Hamiltonian, we incorporate into F_0 the most important terms of R that contribute to ν_G and ν_H. The more terms we include, the better our approximation will be. However, these new terms must not depend on the angles, since we still wish to maintain the integrability of F_0. We will then divide the complete Hamiltonian F as the sum of a (new) unperturbed part F_0 plus a perturbation F_1:

$$F(L,\Lambda,G,H,M,M_1,\omega,\Omega) = F_0(L,\Lambda,G,H) + \varepsilon F_1(L,\Lambda,G,H,M,M_1,\omega,\Omega) \tag{28}$$

where now

$$\begin{aligned} F_0(L,\Lambda,G,H) &= -\frac{\mu^2}{2L^2} + n_1\Lambda - R_0(L,\Lambda,G,H) \\ F_1(L,\Lambda,G,H,M,M_1,\omega,\Omega) &= -R_1(L,\Lambda,G,H,M,M_1,\omega,\Omega) \end{aligned} \tag{29}$$

and R_0 contains the lowest-degree terms (in semimajor axis, eccentricity and inclination) of the disturbing function which do not depend explicitly on any of the angles. Explicitly, we will choose:

$$-R_0(L,\Lambda,G,H) = f_{4,0,0,0,0,0,0}\, L^4 + f_{4,2,0,0,0,0,0}\, L^4 e^2 + f_{4,0,2,0,0,0,0}\, L^4 \xi^2. \tag{30}$$

and the perturbation F_1 (i.e. R_1) contains the remaining terms. Notice that the "small parameter" ε invoked to separate F_1 from F_0 is merely formal, and the real dynamical system corresponds to $\varepsilon = 1$. The actual small parameter is given by $(a/a_1)^2$, which is the difference between the lowest-order terms in F_0 and F_1. However, some terms of order $(a/a_1)^2$ have been brought to the Kernel, and higher order of the semimajor axes ratio also inhabit F_1. For this reason, we $(a/a_1)^2$ is not actually a separator between Kernel and perturbation, and we introduce the formal parameter ε to play this role. It will also help us keep track of the different order in Hori's perturbation series.

The fundamental frequencies obtained from the Kernel F_0 are given by:

$$\begin{aligned}
\nu_{L0} = \frac{\partial F_0}{\partial L} &= \mu^2 L^{-3} + (4 f_{4,0,0,0,0,0,0} + 2 f_{4,2,0,0,0,0,0})\, L^3 + 2 f_{4,2,0,0,0,0,0}\, L^3 e^2 + 4 f_{4,0,2,0,0,0,0}\, L^3 \xi^2 \\
\nu_{\Lambda} = \frac{\partial F_0}{\partial \Lambda} &= n_1 \\
\nu_{G0} = \frac{\partial F_0}{\partial G} &= -2 f_{4,2,0,0,0,0,0}\, L^3 (1-e^2)^{1/2} + \frac{1}{2} f_{4,0,2,0,0,0,0}\, L^3 (1-2\xi^2)(1-e^2)^{-1/2} \\
\nu_{H0} = \frac{\partial F_0}{\partial H} &= \pm \frac{1}{2} f_{4,0,2,0,0,0,0}\, L^3 (1-e^2)^{-1/2}
\end{aligned} \tag{31}$$

In Figure 1 we present two different estimations of the fundamental frequencies. Broken lines correspond to the values obtained from our new F_0, while continuous curves show the results from a first-order averaging of the complete integrable Hamiltonian function, including terms up to fifth-degree in (α, e, ξ). The good qualitative agreement between both sets of curves makes it unnecessary to consider additional terms in the Kernel.

We can now apply Hori's perturbation method to eliminate the mean anomalies M, M_1 from our Hamiltonian. The idea then is to search for a canonical transformation, defined by a Lie-type generating function

$$B = \sum_{i=1}^{\infty} \varepsilon^i B_i(L^*, \Lambda^*, G^*, H^*, M^*, M_1^*, \omega^*, \Omega^*) \tag{32}$$

to new (star) variables such that the new Hamiltonian function

$$F^* = \sum_{i=0}^{\infty} \varepsilon^i F_i(L^*, \Lambda^*, G^*, H^*, \omega^*, \Omega^*) \tag{33}$$

is independent of the pair M^*, M_1^*. In doing this, we will have an integrable approximation of the secular system.

Up to third order in ε, the relationship between the new secular Hamiltonian, the generating function and the old Hamiltonian, is given by:

$$F_0^* = F_0$$

$$
\begin{aligned}
F_1^* &= F_1 + \{F_0, B_1\} \\
F_2^* &= F_2 + \{F_0, B_2\} + \{F_1, B_1\} + \frac{1}{2}\{\{F_0, B_1\}, B_1\} \\
F_3^* &= F_3 + \{F_0, B_3\} + \{F_1, B_2\} + \{F_2, B_1\} + \frac{1}{2}\{\{F_1, B_1\}, B_1\} \\
&+ \frac{1}{2}\{\{F_0, B_1\}, B_2\} + \frac{1}{2}\{\{F_0, B_2\}, B_1\} + \frac{1}{6}\{\{\{F_0, B_1\}, B_1\}, B_1\} \\
&\vdots
\end{aligned}
\tag{34}
$$

where both the right and left hand terms must be written in the new variables, and $\{\ ,\ \}$ is the Poisson bracket. For two arbitrary analytical functions g and h, it is given by:

$$
\begin{aligned}
\{g, h\} &= \left(\frac{\partial f}{\partial M^*}\frac{\partial g}{\partial L^*} - \frac{\partial g}{\partial M^*}\frac{\partial f}{\partial L^*}\right) + \left(\frac{\partial f}{\partial M_1^*}\frac{\partial g}{\partial \Lambda^*} - \frac{\partial g}{\partial M_1^*}\frac{\partial f}{\partial \Lambda^*}\right) + \\
&+ \left(\frac{\partial f}{\partial \omega^*}\frac{\partial g}{\partial G^*} - \frac{\partial g}{\partial \omega^*}\frac{\partial f}{\partial G^*}\right) + \left(\frac{\partial f}{\partial \Omega^*}\frac{\partial g}{\partial H^*} - \frac{\partial g}{\partial \Omega^*}\frac{\partial f}{\partial H^*}\right).
\end{aligned}
\tag{35}
$$

The first (i.e. zero order) equation is trivial; the rest contain two unknowns: the generating function B_i and the new Hamiltonian F_i^*. Choosing F_i^* to be independent of the mean anomalies, we can then use each equation to determine corresponding B_i. This procedure is detailed in the following section.

4.1. The Generating Function

Even though the algebraic system (34) contains an infinite number of equations, each can be solved in terms of the previous orders. For the first onwards, we can write each equation as:

$$
F_n^* = \{F_0, B_n\} + \Phi_n(F_0, \ldots, F_n, B_1, B_{n-1}), \tag{36}
$$

where function Φ_n is assumed to be known from the solution of the previous equations. At each step, we will determine B_n such that F_n^* does not depend explicitly on any angle. We begin by separating Φ_n into two parts:

$$
\Phi_n = \langle \Phi_n \rangle_{M,M_1} + [\Phi_n]_{M,M_1} \tag{37}
$$

where the first sum includes all terms that do not contain the mean anomalies explicitly, and the second sum contains the remaining terms. Introducing this expression into (36), we can solve for both F_n^* and B_n simply choosing:

$$
\begin{aligned}
F_n^* &= \langle \Phi_n \rangle_{M,M_1} \\
-\{F_0, B_n\} &= [\Phi_n]_{M,M_1}.
\end{aligned}
\tag{38}
$$

Although the expression for the new Hamiltonian is explicit, the generating function needs further work. Since $F_0(L^*, \Lambda^*, G^*, H^*)$ is only dependent on the momenta, the Poisson bracket takes the form:

$$
-\{F_0, B_n\} = \nu_L \frac{\partial B_n}{\partial M^*} + \nu_\Lambda \frac{\partial B_n}{\partial M_1^*} + \nu_G \frac{\partial B_n}{\partial \omega^*} + \nu_H \frac{\partial B_n}{\partial \Omega^*}. \tag{39}
$$

Assuming that B_n (and therefore Φ_n) have a Poisson form, we can write:

$$\begin{aligned} -\{F_0, B_n\} &= \sqrt{-1}\sum_{i,j,k}\sum_{l_1,\ldots,l_4} B^{(n)}_{i,j,k,l_1,\ldots,l_4}\ (l_1\nu_L + l_2\nu_\Lambda + l_3\nu_G + l_4\nu_H)\ L^{*i}e^{*j}\xi^{*k} \\ &\times\ E^{\sqrt{-1}(l_1M_1^* + l_2M^* + l_3\omega^* + l_4\Omega^*)}, \end{aligned} \tag{40}$$

where e^* and ξ^* are defined from the canonical G^*, H^* simply inverting the transformations (20). Therefore, we finally obtain from (39):

$$B_n = \sum_{i,j,k}\sum_{l_1,\ldots,l_4} \frac{-\sqrt{-1}\ \Phi^{(n)}_{i,j,k,l_1,\ldots,l_4}}{l_1\nu_L + l_2\nu_\Lambda + l_3\nu_G + l_4\nu_H}\ \ L^{*i}e^{*j}\xi^{*k}\ \ E^{\sqrt{-1}(l_1M_1^* + l_2M^* + l_3\omega^* + l_4\Omega^*)}. \tag{41}$$

From our series expansions for the frequencies, we can rewrite the denominator as:

$$D_{l_1,l_2,l_3,l_4}(L^*) \equiv l_1\nu_L + l_2\nu_\Lambda + l_3\nu_G + l_4\nu_H = l_2\ n_2 + l_1\ \mu\ L^{*-3} + d_2\ L^{*3} \tag{42}$$

where the coefficient d_2 is given by:

$$d_2 = 4l_1\ f_{4,0,0,0,0,0,0} + 2(l_1 - l_3)f_{4,2,0,0,0,0,0} + \frac{1}{2}(l_3 + l_4)f_{4,0,2,0,0,0,0}. \tag{43}$$

Thus, we can rewrite the generating function as:

$$B_n = \sum_{i,j,k}\sum_{l_1,\ldots,l_4} \frac{-\sqrt{-1}\ \Phi^{(n)}_{i,j,k,l_1,\ldots,l_4}}{D_{l_1,l_2,l_3,l_4}(L^*)}\ \ L^{*i}e^{*j}\xi^{*k}\ \ E^{\sqrt{-1}(l_1M_1^* + l_2M^* + l_3\omega^* + l_4\Omega^*)}. \tag{44}$$

The dependence of D_{l_1,l_2,l_3,l_4} with the semimajor axis is not in the Poisson series form. This poses a problem because we wish to retain the self-similarity of all expansions to simplify the calculations of the perturbations series to high orders. To deal with this problem, we expand the inverse of the denominator via a local Taylor series in (L^*, e^*, ξ^*) around a reference value $(L_0^*, 0, 0)$. Concentrating just on the dependence in L^* (the expansions in e^*, ξ^* are analogous) we can write:

$$\frac{1}{D_{l_1,l_2,l_3,l_4}(L^*)} = D^{(0)}_{l_1,l_2,l_3,l_4} + D^{(1)}_{l_1,l_2,l_3,l_4}L^* + D^{(2)}_{l_1,l_2,l_3,l_4}L^{*2} + \ldots \tag{45}$$

where the coefficients are given by:

$$\begin{aligned} D^{(0)}_{l_1,l_2,l_3,l_4} &= D_0^{-1} + D_0^{-2}L_0^*\left(\frac{dD}{dL^*}\right)_{L_0^*} - \frac{1}{2}D_0^{-2}L_0^{*2}\left(\frac{d^2D}{dL^{*2}}\right)_{L_0^*} + D_0^{-3}L_0^{*2}\left(\frac{dD}{dL^*}\right)^2_{L_0^*} \\ D^{(1)}_{l_1,l_2,l_3,l_4} &= -D_0^{-2}\left(\frac{dD}{dL^*}\right)_{L_0^*} + D_0^{-2}L_0^*\left(\frac{d^2D}{dL^{*2}}\right)_{L_0^*} - 2D_0^{-3}L_0^*\left(\frac{dD}{dL^*}\right)^2_{L_0^*} \\ D^{(2)}_{l_1,l_2,l_3,l_4} &= \frac{1}{2}D_0^{-2}\left(\frac{d^2D}{dL^{*2}}\right)_{L_0^*} + D_0^{-3}\left(\frac{dD}{dL^*}\right)^2_{L_0^*} \\ &\vdots \end{aligned} \tag{46}$$

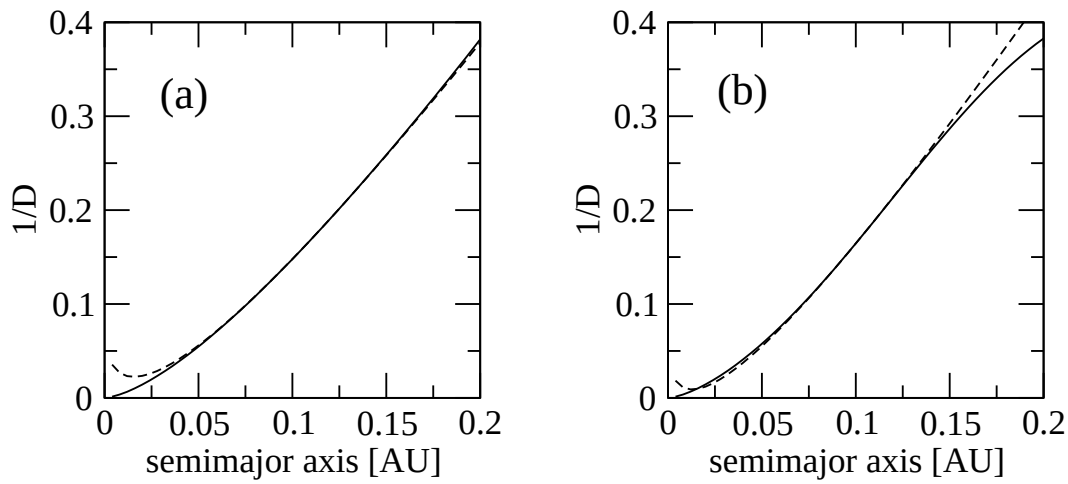

Fig. 2.— Comparison between the inverse of $D_{l_1,l_2,l_3,l_4}(L)$ (continuous lines) and the second-order Taylor expansion in (L, e, ξ) (broken lines), as a function of the semimajor axis. The reference value L_0 was calculated considering $a_0 = 0.1$ AU. **(a)** $l_1 = 1$, $l_2 = 1$, $l_3 = 1$, $l_4 = 1$. **(b)** $l_1 = 1$, $l_2 = -1$, $l_3 = 4$, $l_4 = -4$. In both cases, the eccentricity was chosen $e = 0.2$ and the inclination $I = 10°$. Note that the approximation is good locally near $a_0 = 0.1$ AU.

with $D_0 = D_{l_1,l_2,l_3,l_4}(L_0^*)$, and the derivatives are also evaluated at this value. Introducing the expansion (45) into B_n, we finally obtain the generating function as:

$$B_n = \sum_{i,j,k} \sum_{l_1,\ldots,l_4} B^{(n)}_{i,j,k,l_1,\ldots,l_4} \; L^{*i} e^{*j} \xi^{*k} \; E^{\sqrt{-1}(l_1 M_1^* + l_2 M^* + l_3 \omega^* + l_4 \Omega^*)}, \tag{47}$$

where the new coefficients, up to second degree in $(L^* - L_0^*)$, are given by:

$$B^{(n)}_{i,j,k,l_1,\ldots,l_4} = -\sqrt{-1} \; \left(\Phi^{(n)}_{i,j,k,l_1,\ldots,l_4} \; D^{(0)}_{l_1,l_2,l_3,l_4} + \Phi^{(n)}_{i-1,j,k,l_1,\ldots,l_4} \; D^{(1)}_{l_1,l_2,l_3,l_4} + \Phi^{(n)}_{i-2,j,k,l_1,\ldots,l_4} \; D^{(2)}_{l_1,l_2,l_3,l_4} \right). \tag{48}$$

An example of the precision of this approximation of the denominator of the generating function is shown in Figure 2.

4.2. The Second-Order Secular Hamiltonian

We now have all the tools to determine the secular function $F^*(L^*, \Lambda^*, G^*, H^*)$ up to any given order in ε. In this subsection we will show the explicit expressions for the second-order Hamiltonian $F_0^* + F_1^* + F_2^*$. It has already been mentioned that the zero-order function F_0^* is equal to the original kernel F_0, simply replacing the old variables by the new. For the first order, equations (35) and (37) show that $\Phi_1 = F_1$, so from (36) and (41) we have:

$$F_1^* \;\; = \;\; \sum_{i,j,k} \sum_{l_3,l_4} f_{i,j,k,0,0,l_3,l_4} \; L^{*i} e^{*j} \xi^{*k} E^{\sqrt{-1}(l_3 \omega^* + l_4 \Omega^*)} \tag{49}$$

$$B_1 \;=\; \sum_{i,j,k} \sum_{l_1,\ldots,l_4} B^{(1)}_{i,j,k,l_1,\ldots,l_4} \;\; L^{*i} e^{*j} \xi^{*k} \;\; E^{\sqrt{-1}(l_1 M_1^* + l_2 M^* + l_3 \omega^* + l_4 \Omega^*)},$$

For the second order, the expression become slightly more complicated:

$$\Phi_2 = F_2 + \{F_1, B_1\} + \frac{1}{2}\{\{F_0, B_1\}, B_1\}. \tag{50}$$

However, we do have two advantages. First, due to our choice of perturbation, we have that $F_2 \equiv 0$. Second, from the first-order Hori equation, it can be seen that $\{F_0, B_1\} = F_1^* - F_1$. Introducing both into (50), we obtain:

$$\Phi_2 = \frac{1}{2}\{(F_1^* + F_1), B_1\}. \tag{51}$$

To calculate Φ_2 we have to differentiate B_1 with respect to the momenta. This operation is straightforward and analogous to similar derivatives performed in Section 3.1, since we have kept invariant the Poisson form of each function. Finally, the second-order Hamiltonian is simply:

$$F_2^* = \langle \Phi_2 \rangle. \tag{52}$$

4.3. The Third-Order Secular Hamiltonian

As we shall see in our comparisons with numerical integrations, a second-order theory is not sufficient to obtain a precise model for secular dynamics. Therefore we must calculate the third-order contributions. The expression for F_3^* (see equation (35)) is given by:

$$F_3^* = \{F_0, B_3\} + \Phi_3(F_0, \ldots, F_3, B_1, B_2), \tag{53}$$

where:

$$\begin{aligned} \Phi_3 \;&=\; F_3 + \{F_2, B_1\} + \{F_1, B_2\} + \frac{1}{2}\{\{F_1, B_1\}, B_1\} \\ &+\; \frac{1}{2}\{\{F_0, B_1\}, B_2\} + \frac{1}{2}\{\{F_0, B_2\}, B_1\} + \frac{1}{6}\{\{\{F_0, B_1\}, B_1\}, B_1\}. \end{aligned} \tag{54}$$

Once again, we are able to use some simplifications. First, we have defined $F_2 = F_3 = 0$, thus the first two sums are zero. Second, from the Hori equations of first two orders, we know that:

$$\begin{aligned} \{F_0, B_1\} \;&=\; F_1^* - F_1 \\ \{F_0, B_2\} \;&=\; F_2^* - \frac{1}{2}\{(F_1^* + F_1), B_1\}. \end{aligned} \tag{55}$$

Thus, the expression for Φ_3 simplifies to:

$$\Phi_3 = \frac{1}{2}\{(F_1^* + F_1), B_2\} + \frac{1}{12}\{\{(F_1 - F_1^*), B_1\}, B_1\} + \frac{1}{2}\{F_2^*, B_1\}. \tag{56}$$

Finally, since we are only interested in the secular third-order Hamiltonian, and we will not extend our model beyond this order, we only need the averaged values of Φ_3. Thus,

$$F_3^* = \frac{1}{2}\langle\{F_1, B_2\}\rangle + \frac{1}{12}\langle\{\{(F_1 - F_1^*), B_1\}, B_1\}\rangle. \tag{57}$$

Although we discuss here the Hori series truncated at order 3, our procedure allows a straightforward extension to higher orders. The self-similarity of the expressions allow this extension with no difficulty, except limits dictated by the computer resources (e.g. CPU and RAM restrictions).

5. The Secular Hamiltonian

We obtain a new Hamiltonian function $F^* = F_0^* + F_1^* + F_2^* + F_3^*$ as a function of the new variables $(L^*, G^*, H^*, \omega^*, \Omega^*)$. We call these "mean elements", since they are averaged over the mean anomalies. Since the secular Hamiltonian does not depend explicitly on M, the quantity L^* is a constant of motion of the complete system, and gives the *proper* semimajor axis. The relationship between the starred variables and the original elements is also given by Hori's transformation. Denoting $B = B_1 + B_2$ as the complete (up to second-order) generating function, we can write:

$$\begin{aligned} L &= L^* + \frac{\partial B}{\partial M^*} \quad ; \quad & M &= M^* - \frac{\partial B}{\partial L^*} \\ G &= G^* + \frac{\partial B}{\partial \omega^*} \quad ; \quad & \omega &= \omega^* - \frac{\partial B}{\partial G^*} \\ H &= H^* + \frac{\partial B}{\partial \Omega^*} \quad ; \quad & \Omega &= \Omega^* - \frac{\partial B}{\partial H^*}, \end{aligned} \tag{58}$$

where it is important to remember that B is a function of the starred variables. Thus, if we wish to determine the mean elements from their osculating counterparts, equations (58) must be solved iteratively.

We can now write the the equations of motion for the satellite. We have two choices: either use the Lagrange planetary equations in orbital elements, or the Hamilton equations. Both are equivalent, although the latter have the advantage of allowing further use of perturbation methods.

Recalling that L^* is a constant of motion, and thus a parameter of the Hamiltonian, we have a two degree of freedom system characterized by $F^* = F^*(G^*, H^*, \omega^*, \Omega^*; L^*)$. The equations of motion are:

$$\begin{aligned} \frac{dG^*}{dt} &= \frac{\partial F^*}{\partial \omega^*} \quad ; \quad & \frac{d\omega^*}{dt} &= -\frac{\partial F^*}{\partial G^*} \\ \frac{dH^*}{dt} &= \frac{\partial F^*}{\partial \Omega^*} \quad ; \quad & \frac{d\Omega^*}{dt} &= -\frac{\partial F^*}{\partial H^*}, \end{aligned} \tag{59}$$

where the Hamiltonian is given by:

$$F^* = \sum_{i,j,k}\sum_{l_3,l_4} S_{i,j,k,l_3,l_4} L^{*i} e^{*j} \xi^{*k} E^{\sqrt{-1}(l_3\omega^* + l_4\Omega^*)}. \tag{60}$$

This system can be solved numerically, and its results compared with an exact simulation of Newton's equations. We have chosen a fictitious Jovian satellite with initial conditions given by $a = 0.15$ AU,

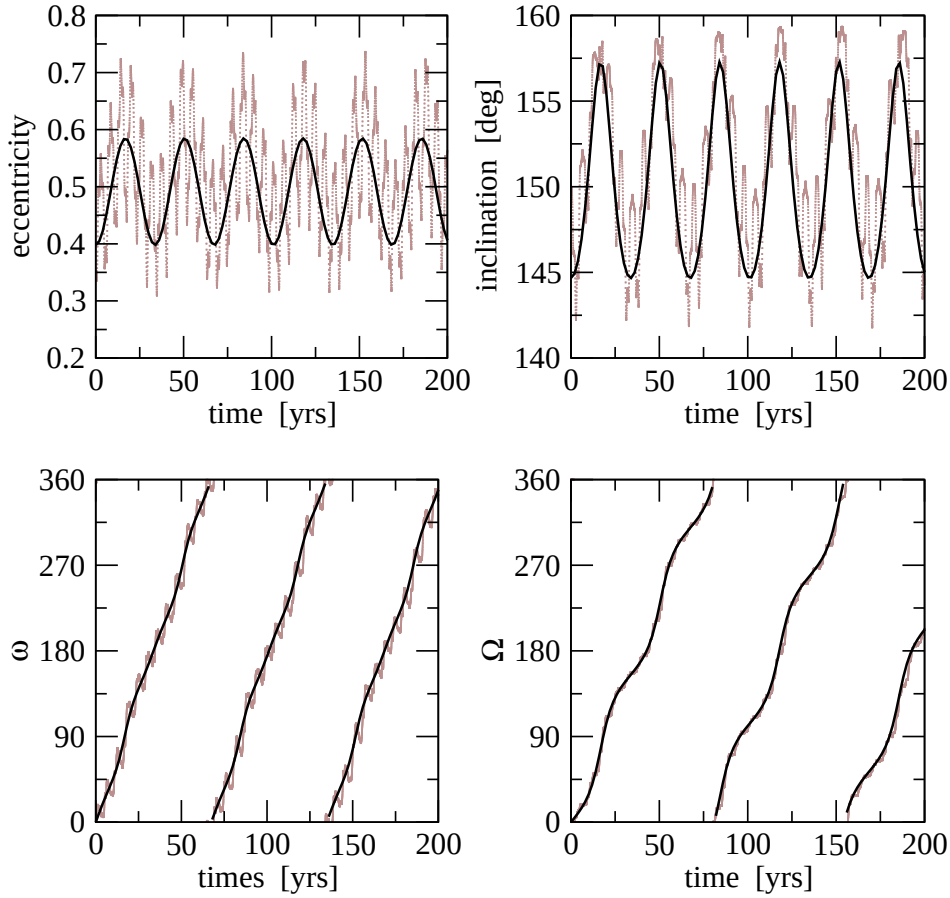

Fig. 3.— Evolution of a fictitious Jovian satellite with the initial orbit defined by $a = 0.15$ AU, $e = 0.37$ and $I = 150$ degrees. All initial angular variables were taken equal to zero. The satellite orbits Jupiter-mass planet is assumed to move in a circular orbit around the Sun at 5.2 AU. Grey dots correspond to the results of an exact numerical simulation (including short-period terms), while continuous black lines show the secular evolution according to our model equations (59).

$e = 0.37$, $I = 150°$ and $M = M_1 = \omega = \Omega = 0$. Jupiter is assumed to move in a circular orbit. Figure 3 shows, in gray dots, the orbital evolution of this body as determined with an exact numerical simulation. In order to apply our secular model, we used the Hori's transformation to pass from the osculating elements to the mean variables. Once the starred variables were determined, we then solved equations (59) numerically. The results are shown in the figure with black continuous lines. We can see a very good agreement with the exact evolution, even though we have considered a highly eccentric and inclined retrograde orbit with large semimajor axis. The periods of oscillation of both angles are also well reproduced. Therefore, our secular Hamiltonian (60) contains all important features of the secular dynamics.

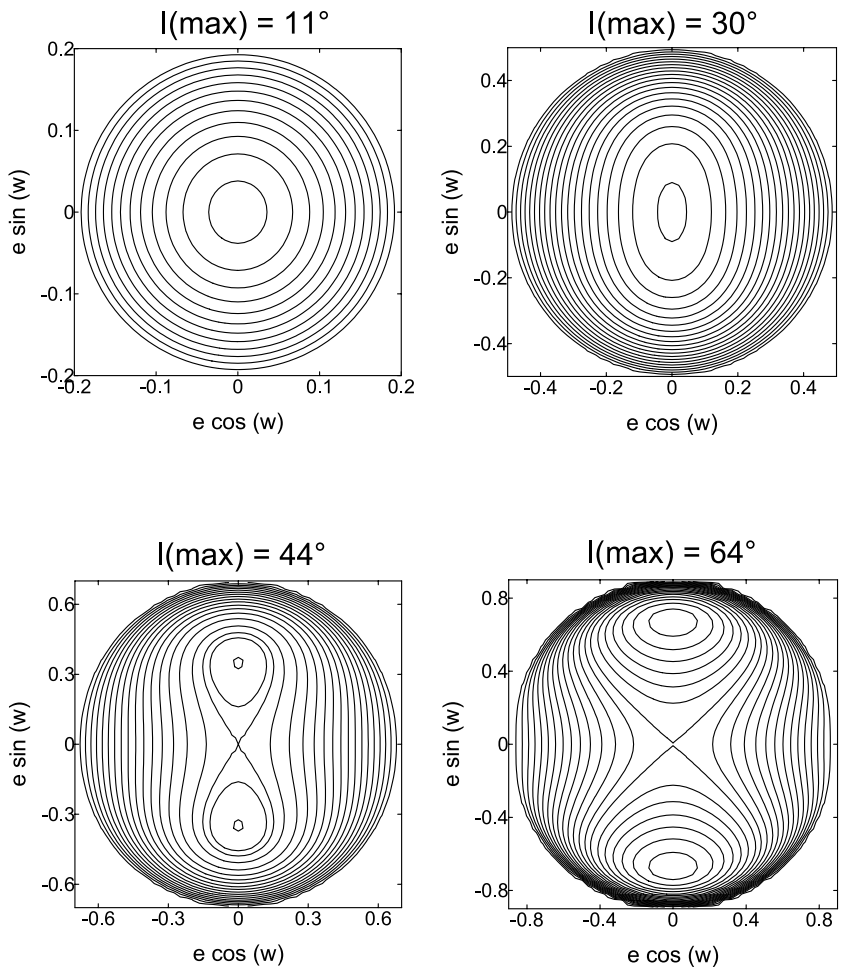

Fig. 4.— Level curves of $F_L = const.$ where F_L defined in (61), for semimajor axis $a = 0.1$ AU and six different values of H. Each is characterized by the maximum value of the inclination (i.e. for $e = 0$). Perturber is Jupiter in a circular orbit with current mass and semimajor axis.

6. The Lidov-Kozai Hamiltonian

The secular system can be further simplified by averaging over the longitude of the node. This can be performed in the same manner as the previous averaging, although we have found that it is usually not necessary to go beyond the first order. We thus search for a new canonical transformation $(G^*, H^*, \omega^*, \Omega^*) \rightarrow (G^{**}, H^{**}, \omega^{**}, \Omega^{**})$ such that the new Hamiltonian (which we will denote by F_L) does not depend explicitly on Ω^{**}. This resulting function is given by:

$$F_L(G, \omega; L, H) = \sum_{i,j,l} K_{i,l}(L, H) e^i \xi^j E^{\sqrt{-1} l \omega} \tag{61}$$

where, for simplicity of notation, we have eliminated the double stars in all the variable. Note that $F_L(G, \omega; L, H)$ is a single degree of freedom system, and both L and H are constant of motions. From equations (20) we

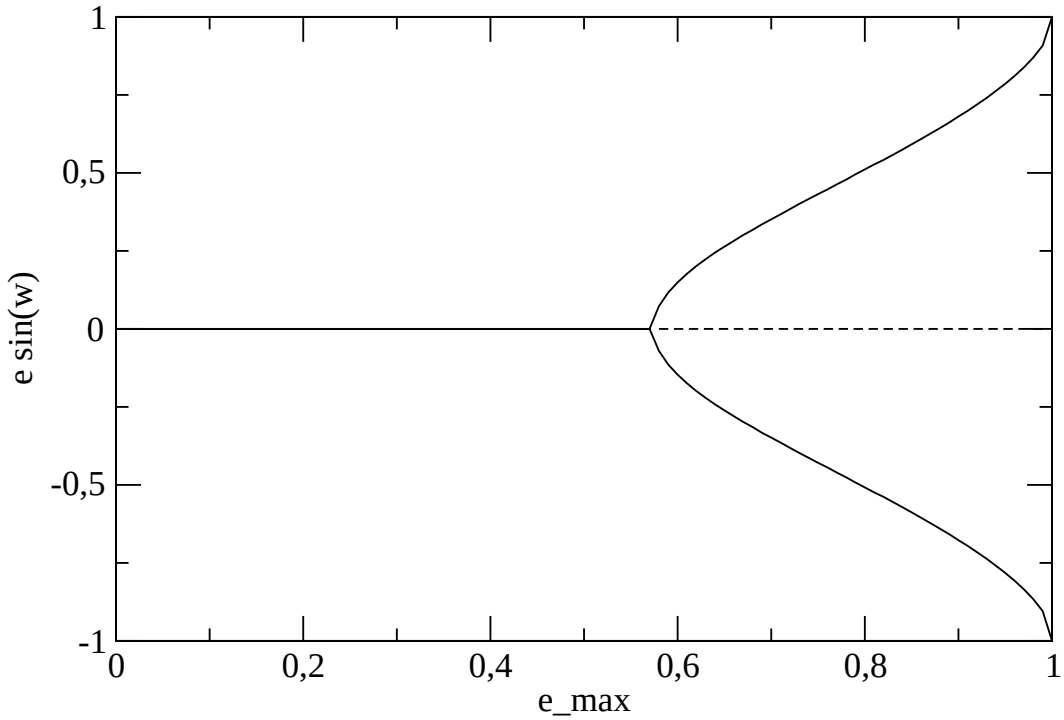

Fig. 5.— Fixed points of F_L for retrograde orbits, as function of the maximum eccentricity, and for $a = 0.1$ AU. Stable solutions are shown by continuous lines, while broken lines correspond to unstable fixed points.

see that:

$$\frac{H}{L} = \sqrt{1 - e^2} \cos I. \tag{62}$$

Since the left hand side is constant throughout the orbital evolution, this means that both the eccentricity and inclination are coupled. As the orbit becomes more inclined, its eccentricity decreases, and e will reach its maximum value for the minimum value of I (or $\pi - I$ for retrograde orbits).

F_L is a high-order version (in ε) of what is usually referred to as the "Kozai Hamiltonian" [1]. The first-order approximation has been extensively studied in the past, both for solar system bodies (e.g. Kozai 1962, Thomas and Morbidelli 1996) and planetary satellites (e.g. Lidov 1961, Nesvorný et al. 2003). Figure 4 shows the level curves of constant values of F_L for retrograde orbits and four different values of H. Each plot is characterized by the value of the maximum value of the inclination (i.e. I_{max}), which is given by equation (62) for $e = 0$:

$$I_{max} = \arccos\left(\frac{H}{L}\right). \tag{63}$$

Jupiter was chosen as the perturber, with current mass and semimajor axis but in a circular orbit.

[1] As recalled recently in Michtchenko et al. (2005), the dynamical phenomena associated to this one degree-of-freedom system, including the so called Kozai resonance, was first discovered by Lidov (1961)

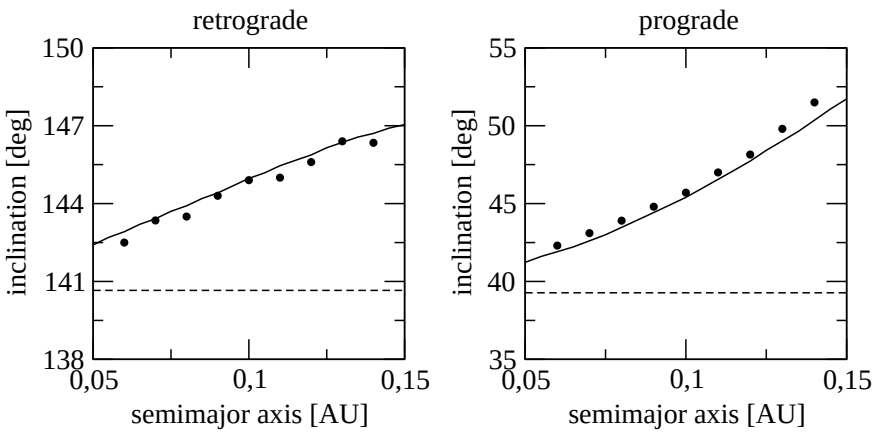

Fig. 6.— Bifurcation value of the proper inclination for the Lidov-Kozai Hamiltonian, as function of the proper semimajor axis and initial circular orbits. Result for retrograde motion are presented on the left-hand plot, while those for direct orbits are shown on the right-hand graph. Dotted lines are results from the "classical" low-order Lidov-Kozai Hamiltonian. Continuous lines show results from our model. Full circles correspond to data from numerical integrations.

We can see the well-known structure of the phase plane. For low values of e_{max} (i.e. low eccentricities and quasi-planar orbits) the level curves are distorted ellipses around the center, which corresponds to a stable fixed point. For $e_{max} \approx 0.58$, the origin becomes unstable and bifurcates into two new stable points with $\omega = \pm 90°$. A separatrix appears, dividing the phase plane into a region of circulatory motion (now restricted to high values of $e \sin \omega$) and two libration islands. This structure is known as the Kozai resonance.

Figure 5 shows the fixed points of the Hamiltonian (retrograde orbits), as well as their stability as function of e_{max}. Stable solutions are shown by continuous lines, while unstable fixed points are identified by broken curves. The bifurcation is clearly visible for $e_{max} \approx 0.58$, and the eccentricity of the center of the Lidov-Kozai resonance tends towards unity with the value of e_{max}.

The "classical" expression for the Lidov-Kozai found in many papers (e.g. Kinoshita and Nakai 1999, Carruba et al. 2002, Nesvorný et al. 2003) is given by:

$$K = \frac{m_0 n_1^2 a_1^2}{16(m_0 + m_1)} \left((2 + 3e_1^2)(3\cos^2 I_1 - 1) + 15 e_1^2 \sin^2 I_1 \cos 2\omega_1 \right), \tag{64}$$

and is obtained retaining only the lowest order terms in semimajor axis, eccentricity and inclination, and performing the averaging over the mean anomalies only to first order. The bifurcation points of this Hamiltonian are independent of the semimajor axis, and direction of the orbital motion of the satellite. However, the empirical model of Cuk and Burns (2004) shows that higher-order perturbations introduce a dependence of e_{max} with the semimajor axis.

Figure 6 shows the application of our model. Result for retrograde orbits are presented on the left-hand plot, while those for direct orbits are shown on the right-hand graph. Both show the corresponding values of the inclination for which the libration zone first appears for initially circular orbits. Dotted lines

are results from the "classical" low-order Lidov-Kozai Hamiltonian. Continuous lines show results from our model. Full circles correspond to data from numerical integrations. Even for large semimajor axes the overall agreement is very good, with differences which rarely exceed ~ 0.5 degrees. However, we do note a tendency for the model to underestimate (overestimate) the numerical values by for retrograde (direct) motion.

7. Secular Frequencies of Fictitious Satellites

A very important application of a secular theory for the irregular satellites of the outer planets involves the determination of the frequencies of oscillation of both the argument of pericenter (i.e. ν_G) and the longitude of the ascending node (i.e. ν_H). Having expression for these quantities (explicit or otherwise) as function of the initial orbital elements, we can then search for the regions of secular resonances with the fundamental frequencies of the outer planets and, in a future work, construct a model for each of these secular commensurabilities.

We have developed two different methods for the calculation of the secular frequencies. Each is detailed below:

- *Full Averaging*: The elimination of the angular variables described in Section 4 is extended to also include the secular angles ω and Ω. This direct elimination of the argument of pericenter is equivalent to assuming that the origin $(G\cos\omega, G\sin\omega)$ is an elliptic fixed point and, therefore, the angle ω circulates. In other words, we assume that we are far from the Lidov-Kozai resonance. Since the final expression depends only on the momenta, the temporal evolution of the starred variables are given by:

$$\begin{aligned} L^*(t) &= const. \;\; ; \;\; M^*(t) = \nu_{L^*} t + M_0^* \qquad \text{where} \;\; \nu_{L^*} = \frac{\partial F^*}{\partial L^*} = const. \\ G^*(t) &= const. \;\; ; \;\; \omega^*(t) = \nu_{G^*} t + \omega_0^* \qquad \text{where} \;\; \nu_{G^*} = \frac{\partial F^*}{\partial G^*} = const. \\ H^*(t) &= const. \;\; ; \;\; \Omega^*(t) = \nu_{H^*} t + \Omega_0^* \qquad \text{where} \;\; \nu_{H^*} = \frac{\partial F^*}{\partial H^*} = const. \end{aligned} \tag{65}$$

  and with the total Hamiltonian $F^* = F_0^* + F_1^* + F_2^* + F_3^*$. It is clear that (L^*, G^*, H^*) are integrals of motion of this approximate model, and the corresponding angles change linearly with time. The values of ν_{L^*},ν_{G^*},ν_{H^*} are sometimes referred to as the proper frequencies of the system. Finally, applying the transformation from canonical momenta to orbital elements yields constant values of (a^*, e^*, I^*) which constitute a set of proper orbital elements of each solution. Although there are several different definitions of proper elements, the present one has the advantage of representing the averaged temporal values of each orbital element, instead of (for example) their maximum of minimum for given values of the angular variables.

  Equations (65) and (58) have three important applications. First, from the initial conditions given in osculating (i.e. non-starred) variables, we can solve these equations iteratively to obtain the proper elements (a^*, e^*, I^*) and the proper frequencies. Second, from these proper elements, we can then

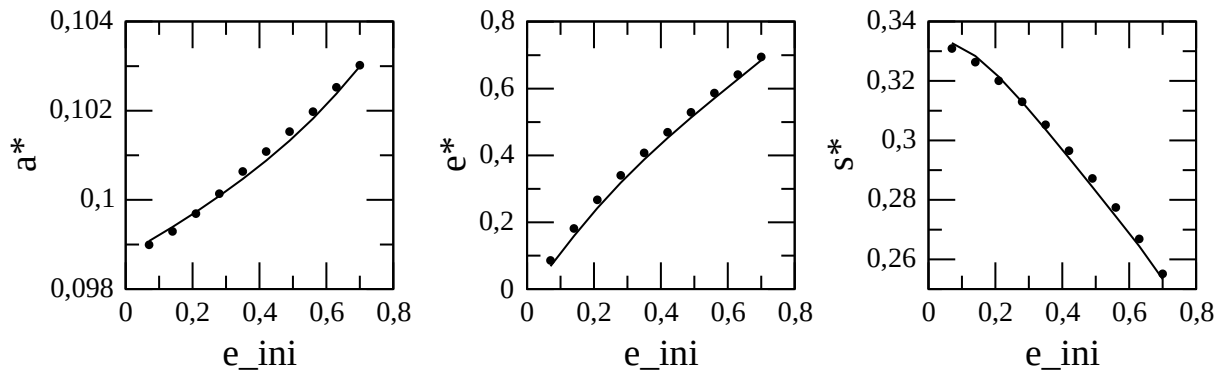

Fig. 7.— Numerical proper elements (filled circles) and analytical values obtained with a full averaging (continuous lines), as function of the initial eccentricity. Other initial orbital values were taken equal to $a_{ini} = 0.1$ AU, $I_{ini} = 40$ deg. and $\omega_{ini} = \Omega_{ini} = 0$. Recall that $s = \sin(I/2)$.

deduce the maximum and minimum value of the original osculating elements, or their value for any given set of the angles. This allows us to transform from one definition of proper element to any other, thus simplifying a comparison with other works, especially those numerical in nature. Finally, these solutions represent the secular dynamics of the system, and are valid as long as we are not in the vicinity of any secular (or Lidov-Kozai) resonance.

- *Partial Averaging & Lidov-Kozai*: If the eccentricity and/or inclination of the satellite is very large, the structure of the secular phase plane defined by the Lidov-Kozai resonance cannot be neglected, and the full averaging yields imprecise results. In such a case, we must limit the averaging in Section 4 to the mean anomalies, and solve the resulting Lidov-Kozai Hamiltonian in full.

7.1. Comparisons with Numerical Integrations

To test both these approaches, we have integrated 1452 test satellites for 10^5 years. The particles were placed around a Jupiter-like planet on a circular orbit with $a = 5.2$ AU. The mass of the planet was assumed to be 10^{-3} that of the Sun. The satellite orbits were chosen within the following intervals: $a = 0.05 - 0.15$ AU, $e = 0 - 0.7$, $I = 0 - 50$ degrees (for direct orbits) and $I = 130 - 180$ degrees for the retrograde case. The initial longitudes were assumed to be zero. The initial configuration was such that the Sun, the planet and the satellite are on the x-axis, and the satellite was in the pericenter and the ascending node of its orbit.

The results of the numerical integrations were Fourier analyzed using 16384 points separated by 5 years, yielding a total time span of 81920 years. Due to the usual confusion about the definition of the longitude of pericenter for retrograde orbits, we determined the frequencies of the nodal longitude and the argument of the pericenter instead. The argument of the pericenter ω is the angle between node and pericenter, and is always positive by definition. Thus, $\varpi = \Omega + \omega$ for prograde orbits and $\varpi = \Omega -$

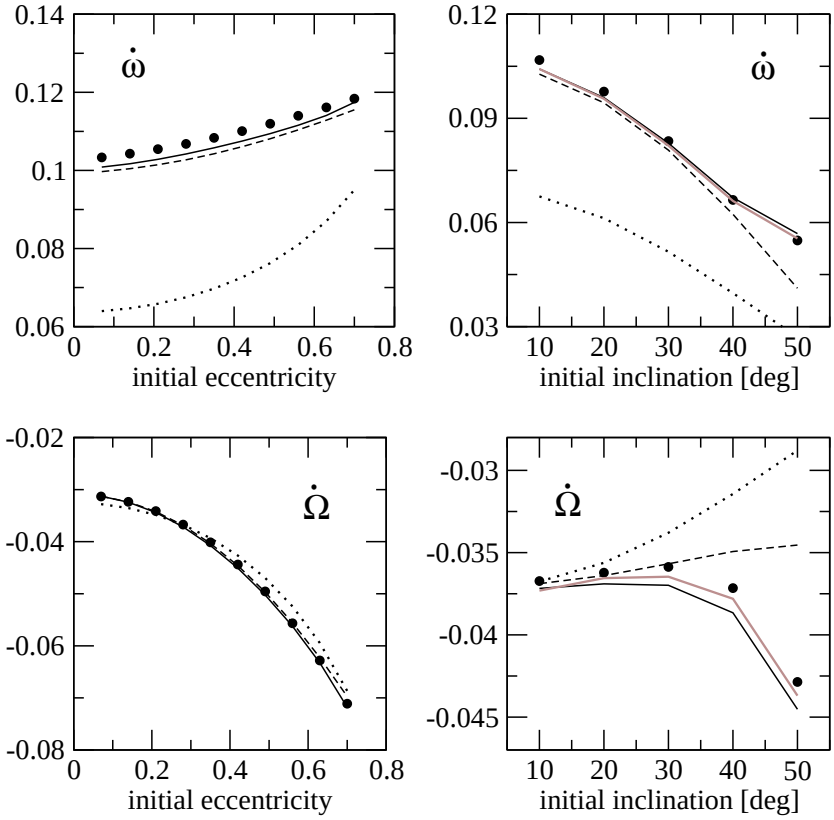

Fig. 8.— Frequencies of the pericentric argument ν_G and of the node ν_H for fictitious Jovian direct satellites. **Left:** Frequencies as a function of the initial eccentricity, for $a_{ini} = 0.1$ AU and $I_{ini} = 10$ deg. **Right:** Frequencies as a function of the initial inclination for $a_{ini} = 0.1$ AU and $e_{ini} = 0.28$. Numerical data is presented as filled circles, while the analytical results are shown in lines. The dotted lines show the frequencies obtained from the first-order model; dashed line represents the second-order model, and the continuous line the third-order. On the right-hand plots, gray lines correspond to results obtained via a partial averaging (over the mean anomalies) and a semi-numerical solution of the Lidov-Kozai Hamiltonian.

ω for retrograde orbits. The proper eccentricities and inclinations were estimated as the largest Fourier-term amplitude in $(e\cos\omega, e\sin\omega)$ and in $(I\cos\Omega, I\sin\Omega)$, using the Fourier method by Sidlichovský and Nesvorný (1987). The proper semimajor axis was take equal to its mean value over the total integration time span.

Figure 7 shows an example of the estimation of proper orbital elements. Filled circles are the numerical results from our simulations, while the continuous lines are the analytical values calculated from our full averaged third-order model. We can see a very good agreement, even though the definition of the numerical and analytical proper elements is not exactly the same. In Figure 8 we show the calculations of the frequencies of the pericentric distance (i.e. ν_G) and of the ascending node (i.e. ν_H) for a series of fictitious direct Jovian satellites. As mentioned in Cuk and Burns (2004) (see also Saha and Tremaine 1993) in the case of

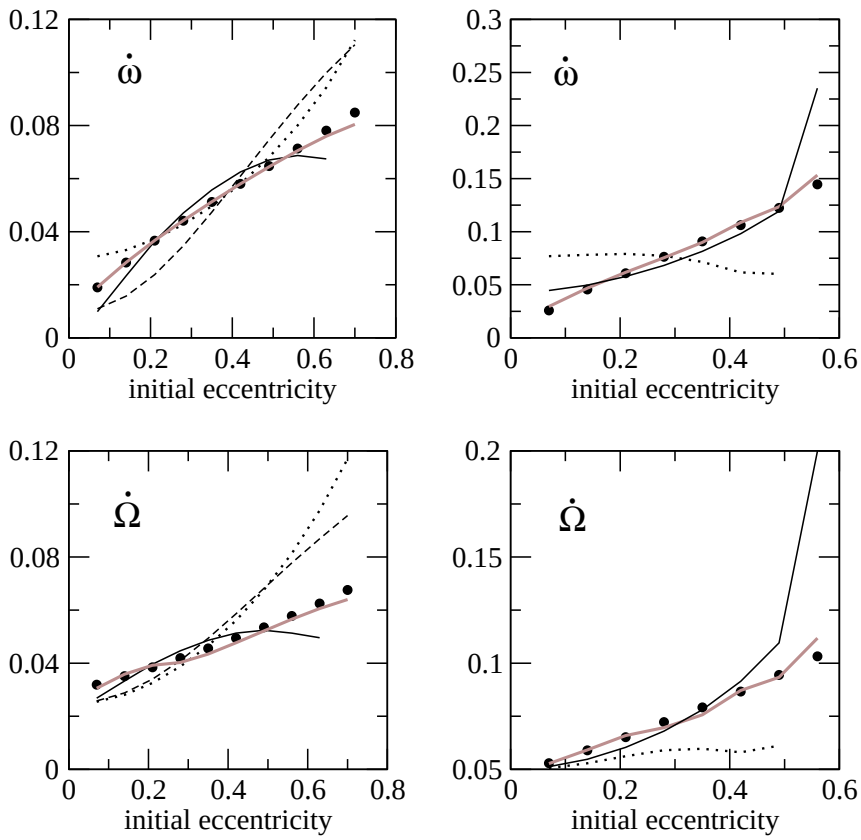

Fig. 9.— Frequencies of the pericentric argument ν_G and of the node ν_H for fictitious Jovian retrograde satellites. **Left:** Frequencies as a function of the initial eccentricity, for $a_{ini} = 0.1$ AU and $I_{ini} = 140$ deg. **Right:** Same as left plots, but for $a_{ini} = 0.15$ AU and $I_{ini} = 150$ deg. The meaning of the different types of curves is the same as for the previous figure.

ν_G we note a big difference between the first-order and second-order averaging. For low inclinations the nodal rate can be successfully approximated by low-order solutions.

The right-hand plots of Figure 8, especially for high inclinations, show the importance of the Lidov-Kozai resonance. Although none of these fictitious satellites are actually in a ω-libration, the change in the topology of the phase plane makes the full averaging much more imprecise than a correct treatment of the Lidov-Kozai Hamiltonian.

Figure 9 shows similar results, but now for retrograde orbits with high inclinations with respect to the invariant plane. Recall also that since the initial eccentricity is taken as the value of e for $\omega = 0$, this initial value actually corresponds to the minimum eccentricity of the orbit. Thus, the mean and maximum values attained by this orbital element throughout its evolution is actually much higher. On the right-hand side, corresponding to initial semimajor axis equal to $a_{ini} = 0.15$ AU, the second-order full averaging yields frequencies which appear outside the range of the graphs, and are thus completely unreliable.

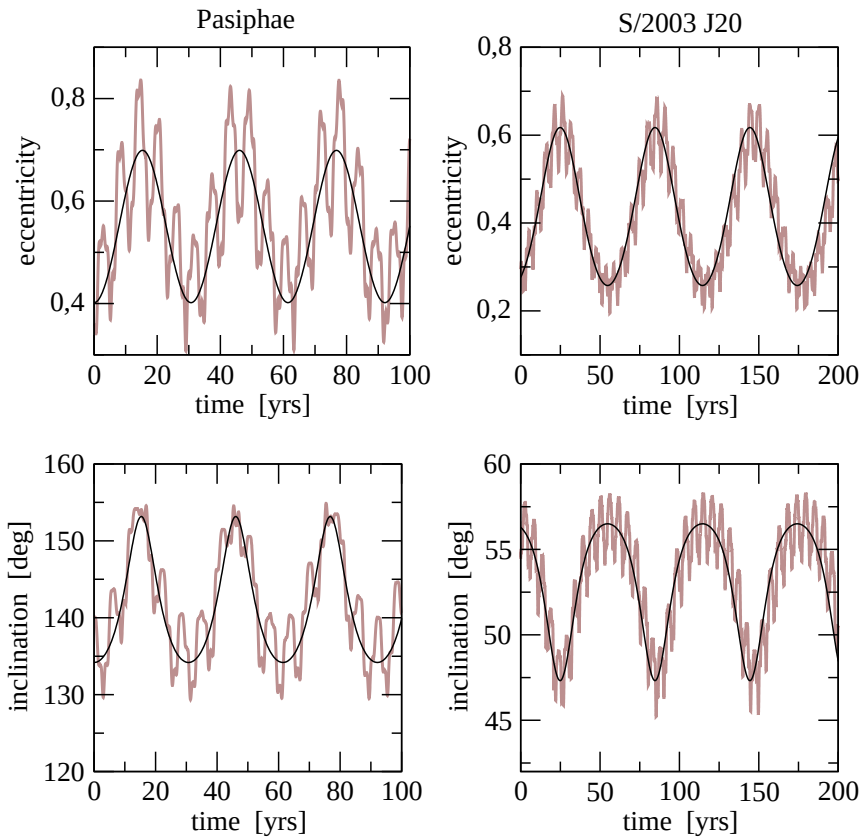

Fig. 10.— Numerical simulation (gray) versus third-order secular model (black) showing the orbital evolution of the satellites Pasiphae and S/2003 J20, considering the planet in a circular and planar orbit.

7.2. Application to Real Satellites

As a final test, we apply our model to two particularly difficult cases: Pasiphae and Nereid. The first is an irregular satellite of Jupiter with high inclination and large eccentricity. The second is the well known Neptune's moon which moves in a nearly planar, highly eccentric orbit. The initial values of the semimajor axis, eccentricity and inclination adopted for our study were as follows. For Pasiphae: $a = 0.156$ AU, $e = 0.379$ and $I = 140.1$ degrees. For Nereid, $a = 0.037$ AU, $e = 0.75$ and $I = 5.04$ degrees. In both cases, the perturber was assumed to move in a planar and circular orbit.

Figure 10 shows the evolution of the eccentricity and inclination for a few periods of the secular angles. Left hand plots correspond to Pasiphae, while the right-hand graphs show results for the recently discovered Jovian satellite S/2003 J20. In gray we present the orbital variation obtained from an exact numerical integration of the three-body problem. In black we show the results of the secular model. Note that in both cases the agreement is very good, not only with regards to the frequencies of the secular angles, but also in the amplitudes of oscillation. For S/2003 J20 a small discrepancy is noted for the secular frequencies, of the order of 1%; however this satellite has a large semimajor axis and inclination. Moreover, this particular

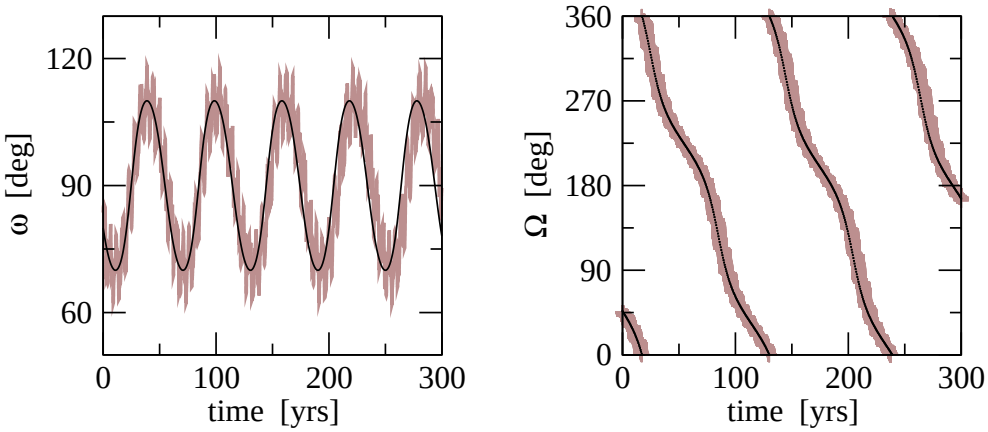

Fig. 11.— Jovian satellite S/2003 J20: temporal evolution of the argument of pericenter (left) and longitude of the ascending node (right). Grey lines correspond to direct numerical simulations while black indicate the results of our third-order secular model. Note the libration of ω around 90 degrees.

body is located deep inside the Lidov-Kozai resonance. In Figure 11 shows the evolution of the argument of the pericenter (left) and longitude of the ascending node (right). Grey curves show the results of a direct numerical simulations, while our analytical results are presented in black. The libration of ω around 90 degrees is very well reproduces, both in amplitude and period of oscillation.

8. Conclusions

We have described here a new high-order analytical model for the secular (i.e. long-term) orbital evolution of planetary satellites. The model is based on a high-order Legendre expansion of the disturbing Hamiltonian and a third-order Hori perturbation theory. In this paper we have included all terms in the Hamiltonian up to fifth-order in a_1/a_2, e_1 and η (or ξ).

The model is valid for any inclination (prograde or retrograde) and eccentricities below ~ 0.67. Practically all known irregular satellites of the outer planets lie within this convergence limit. The only exceptions are Pasiphae, S/2000 S4 and Setebos (all with $e_{max} \sim 0.71$) and Nereid ($e_{max} \sim 0.75$). However, as has been shown above, even in these cases the precision of the third-order truncated series is very good.

Comparisons with exact numerical integrations have shown that the model yields accurate results even for high eccentricities and inclinations, and large separations of a satellite from the parent planet. We have used this model to briefly analyze the phase space in the vicinity of the irregular satellites of the outer planets. We found that the Kozai resonance occurs at progressively larger (proper) inclinations with increasing separation of the satellite from the parent planet. We also calculated the precession frequencies of orbits of the irregular satellites at Jupiter, These results will be particularly useful for determining the locations and strengths of secular resonances in the space occupied by distant satellite orbits.

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REFERENCES

Brouwer, D. and Clemence, G.M. (1961). “Methods of Celestial Mechanics”, Academic Press Inc., London.

Burns, J.A. (1986). in “Satellites”, ed. J.A. Burns and M.S. Matthews, Tucson Univ. Arizona Press, pp 1.

Carruba, V., Burns, J.A., Nicholson, P.D. and Gladman, B.J. (2002). On the inclination distribution of the Jovian irregular satellites. *Icarus*, **158**, 434-449.

Cuk, M. and Burns, J.A. (2004). On the secular behavior of irregular satellites. *AJ*, **128**, 2518-2541.

Delaunay, C.E. (1860). Théorie du mouvement de la Lune I. *Mém. Acad. Sci. Paris*, **28**, 1-883.

Delaunay, C.E. (1867). Théorie du mouvement de la Lune II. *Mém. Acad. Sci. Paris*, **29**, 1-931.

Ferraz-Mello, S. (1994). The convergence domain of the Laplacian expansion of the disturbing function. *Cel. Mec. & Dyn. Astr.*, **58**, 37-52.

Henrard, J. (1989). A survey of Poisson series processors. *Celest. Mech.*, **45**, 245-253.

Hori, G. (1966). Theory of general perturbations with unspecified canonical variables. *Pub. Astron. Soc. Japan*, **18**, 287.

Hughes, S. (1981). The computation of tables of Hansen coefficients. *Celest. Mech.*, **25**, 101-107.

Kaula, W.M. (1961). Analysis of gravitational and geometric aspects of geodesic utilization of satellites. *Geophys. J.*, **5**, 104-133.

Kaula, W.M. (1962). Development of the lunar and solar disturbing functions for a close satellite. *Astron. J.*, **67**, 300-303.

Kinoshita, H. and Nakai, H. (1991). Secular perturbations of fictitious satellites of Uranus. *Cel. Mech. & Dyn. Astr.*, **52**, 293-303.

Kozai, Y. (1962). Secular perturbations of asteroids with high inclination and eccentricity. *Astron. J.*, **67**, 591-598.

Lee, M.H. and Peale, S.J. (2003). Secular evolution of hierarchical planetary systems. *ApJ*, **592**, 1201-1216.

Lidov, R. (1961). Analiz evolucii orbit iskustevennich sputnikov. *Problemi dvigenia iskustvennich nebesnich tel*, Izd. Akademii Nauk SSSR, Moscow (1993), 119-134.

Milani, A. and Knežević, Z. (1994). Asteroid proper elements and the dynamical structure of the asteroid main belt. *Icarus*, **107**, 219-254.

Michtchenko, T.A., Ferraz-Mello, S. and Beaugé, C. (2005). 3-D secular dynamics of the Ups Andromeda planetary system. *ApJ*, submitted.

Murray, C.D. and Dermott, S.F. (1999). "Solar System Dynamics", Cambridge University Press, Cambridge.

Nesvorný, D., Alvarellos, J.L.A., Dones, L. and Levison, H. (2003). Orbital and collisional evolution of the irregular satellites. *Astron. J.*, **126**, 398-429.

Saha, P. and Tremaine, S. (1993). The orbits of the retrograde Jovian satellites. *Icarus*, **106**, 549-562.

Sidlichovský, M. and Nesvorný, D. (1997) Frequency modified Fourier transform and its application to asteroids. *Cel. Mech. & Dyn. Astr.*, **65**, 137-148.

Thomas, F. and Morbidelli, A. (1996). The Kozai resonance in the outer solar system and the dynamics of long-period comets. *Cel. Mech. & Dyn. Astr.*, **64**, 209-229.

Touma, J. and Wisdom, J. (1998). Resonances in the early evolution of the Earth-Moon system. *AJ*, **115**, 1653-1663.

Von Zeipel, H. (1916). Reserches sur le mouvement des petites planètes. *Arkiv. Math. Astron. Fysik*, **11**, 1.

Wintner, A. (1941). "The Analytical Foundations of Celestial Mechanics", Princeton University Press, Princeton.

Yokoyama, T., Santos, M.T., Cardin, G. and Winter, O.C. (2003). On the orbits of the outer satellites of Jupiter. *A&A*, **401**, 763-772.

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