OBLIQUITY VARIATIONS IN PLANETARY SYSTEMS. Wm. R. Ward, C. B. Agnor, and R. M. Canup, *Department of Space Studies, Southwest Research Institute, 1050 Walnut Street, Suite 426, Boulder CO 80302*.

The stability of spin axis motions and the consequent planetary obliquities for a given planetary system is examined via a simple algorithm for spin axis oscillations of a test particle as a functions of its spin rate and orbital distance. For any specified planetary system with masses M_j, orbital distances a_j, eccentricities e_j inclinations I_j, longitudes of periapses and ascending nodes, ϖ_j, Ω_j the Laplace-Lagrange first-order secular solution is determined [1]. This gives the time variation for each planet

$$\begin{aligned}\{h_j, k_j\} &\equiv e_j\{\cos,\sin\}\varpi_j\\ &= \sum_k A_{j,k}\{\cos,\sin\}(s_k t+\epsilon_k)\\ \{p_j, q_j\} &\equiv \sin I_j\{\cos,\sin\}\Omega_j\\ &= \sum_k B_{j,k}\{\cos,\sin\}(s'_k t+\delta_k)\end{aligned}$$

where $\{A_{j,k},\epsilon_k\}$, $\{B_{j,k},\delta_k\}$ are found from the initial conditions at time $t=0$. The variables $\{s_k, s'_k\}$ are the eigenfrequencies of the system.

The orbit evolution of a test object is then given by

$$\{\dot h, \dot k\} = 2an[\Sigma_j\mu_j N_j\{k,-h\} - \Sigma_j\mu_j P_j\{k_j,-h_j\}]$$

$$\{\dot p, \dot q\} = 2an[\Sigma_j\mu_j N_j\{-q,p\} - \Sigma_j\mu_j N_j\{-q_j,p_j\}]$$

where n is its mean motion and

$$N_j \equiv \frac{1}{8a_>}\alpha_j b^{(1)}_{3/2}(\alpha_j);\ P_j \equiv \frac{1}{8a_>}\alpha_j b^{(2)}_{3/2}(\alpha_j)$$

with $\{a_>, a_<\} \equiv \{\max,\min\}(a,a_j)$, $\alpha_j \equiv a_</a_>$, and $b_s^{(m)}(\alpha_j)$ is the so-called Laplace coefficient [1]. The quantities μ_j are the planetary masses normalized to the primary's mass. Substituting for h_j, k_j, p_j, q_j, reversing the order of summation, and integrating gives,

$$\begin{aligned}\{h,k\} &= A_o\{\sin,\cos\}(gt+\epsilon)\\ &\quad +\Sigma_k\left(\frac{g_k}{g-s_j}\right)\{\sin,\cos\}(s_k t+\epsilon_j)\\ \{p,q\} &= B_o\{sin,cos\}(-gt+\delta)\\ &\quad +\Sigma_k\left(\frac{g'_k}{g+s'_k}\right)\{\sin,\cos\}(s_k t+\delta_k)\end{aligned}$$

where,

$$\begin{aligned}g(a) &\equiv 2n\Sigma_j\mu_j(aN_j)\\ g'_k(a) &\equiv 2n\Sigma_j\mu_j(aN_j)B_{j,k}\\ g_k(a) &\equiv 2n\Sigma_j\mu_j(aP_j)A_{j,k}\end{aligned}$$

The integration constants A_o, B_o, ϵ, δ are determined by the initial orientation of the test particle's orbit. The first two quantities are respectively its free eccentricity and the sine of the free inclination.

The next step is to compute the obliquity variation of the test object that results from the orbit evolution. Ward [2,3] provided a first-order solution for Mars that is easily adapted to our problem,

$$\Delta\theta = \frac{gB_o}{\alpha\cos\theta - g}\sin[(\alpha\cos\theta - g)t + \tilde\delta]$$
$$-\Sigma_k\left(\frac{s'_k}{\alpha\cos\theta + s'_k}\right)\left(\frac{g'_k}{g+s'_k}\right)\sin[(\alpha\cos\theta + s'_k)t+\tilde\delta_k]$$

where $\tilde\delta_k$, $\tilde\delta$, are phases found from δ_k and the initial spin axis position of the test planet [2]. In the above expression, α is the spin precession variable given by

$$\alpha \equiv \frac{3\pi}{P}\left(\frac{D}{P}\right)\frac{J_2}{K}(1-e^2)^{-3/2}$$

where P, D are the length of the planet's year and day, J_2 is the second harmonic of its gravity field, and $K = C/MR^2$ is its gyration constant [4]. Because the various eigenfrequencies of the orbit are uncorrelated, they can on occasion constructively interfere. Accordingly, the maximum obliquity variation is

$$\begin{aligned}|\Delta\theta|_{max} &= \left|\frac{gB_o}{\alpha\cos\theta - g}\right|\\ &\quad +\Sigma_k\left|\left(\frac{s'_k}{\alpha\cos\theta+s'_k}\right)\left(\frac{g'_k}{g+s'_k}\right)\right|\end{aligned}$$

Note that singularities exist whenever any of the following conditions occur:

$$g = -s'_k \quad ; \quad \alpha\cos\theta = -s'_k \quad ; \quad \alpha\cos\theta = g$$

The first singularity condition locates the semi-major axes where there is a secular *orbit-orbit* resonance between the test planet's orbit and an eigenfrequency of the planetary system; these are vertical lines, independent of the planet's rotation rate. The obliquity amplitude increases as these resonances are approached because the forced inclination of the orbit goes up. The second condition traces out the length of day vs. semi-major axis of secular *spin-orbit* resonances for each eigenfrequency. Because $J_2 \propto 1/D^2$, $\alpha \propto 1/D$, the curves monotonically decrease [4], *i.e.*, $D \propto 1/P^2 \propto 1/a^3$, and this trend is easily discernable in the figure below. The third condition traces out the secular spin-orbit resonances of the free nodal precession of the test planet; it is not monotonic because the value of $g(a)$ peaks in the vicinity of each planet in the system. The superposition of the obliquity contribution around each of the singularity curves results in complex stability zones.

We have automated this process to produce color coded maps of the obliquity "topography" for an arbitrary planetary system. An example output is shown in Figure 1 where we have first removed Mars from our solar system and solved for the Laplace-Lagrange secular system in its absence (Table 1). Mars is then replaced with a massless test object of similar attributes, viz., average density 3.9 g/cm^3, gyration constant $K = 0.366$, unperturbed obliquity of $\theta = 25^o$, and located in a circular orbit with 2.9^o free inclination. Hydrostatic equilibrium is assumed, with J_2 being estimated by the Darwin-Radau relationship [4]. This procedure allows us to move the pseudo-Mars object without having to construct a new secular system for each location. The linear range $|\Delta\theta|$ of obliquity variations in the inner solar system is displayed as a function of its rotation period and semi-major axis. (The range of variation is twice the value shown, *i.e.* $\pm|\Delta\theta|$.) The current location of Mars on the diagram (24.6 hrs, 1.52 AU) places it in a region where obliquities vary by $\pm\mathcal{O}(10^o)$ which is consistent with recent numerical experiments [5, 6]. This is bounded on the outside by a secular orbit-orbit resonance with s_5' and on the inside by a secular spin-orbit resonance with s_3', so that a slight displacement inward or outward brings the test object into a region of greatly increased oscillations. A niche where Earth-like stability would be exhibited (*i.e.*, $|\Delta\theta| \sim 1.5^o$) is difficult to find inside $\sim$ 2.5AU. The Earth owes its spin axis stability to the lunar torque, which decreases its precession period from 8.1×10^4 to 2.6×10^4 years [2,7]. This has roughly the same effect as lowering the rotation period by a factor of 3. However, a similar satellite contribution to Mars' precession would not be stabilizing, but would instead force it deeper into the $k = 3$ spin-orbit resonance.

TABLE 1: INCLINATION EIGENFREQUENCIES.

	With Mars ("/yr)	*Without Mars* ("/yr)
s_1'	-5.23	-5.17
s_2'	-6.60	-6.49
s_3'	-18.81	-18.49
s_4'	-17.69	——-
s_5'	-25.51	-25.51
s_6'	-2.92	-2.92
s_7'	-0.68	-0.68

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REFERENCES

[1] Brouwer and Clemence, 1961. Methods of Celestial Mechanics, (New York Academic Press). [2] Ward, 1974. J.G.R. **79**, 3375-3386. [3] Ward, 1979. J.G.R. **84**, 237-241. [4] Kaula, *An Introduction to Planetary Physics*, (John Wiley & Sons, Inc., New York). [5] Ward & Rudy 1991. *Icarus* **94**, 160. [6] Touma & Wisdom 1993. *Science* **259**, 1294-1297. [7] Ward, 1982. *Icarus*, **50**, 444-448.

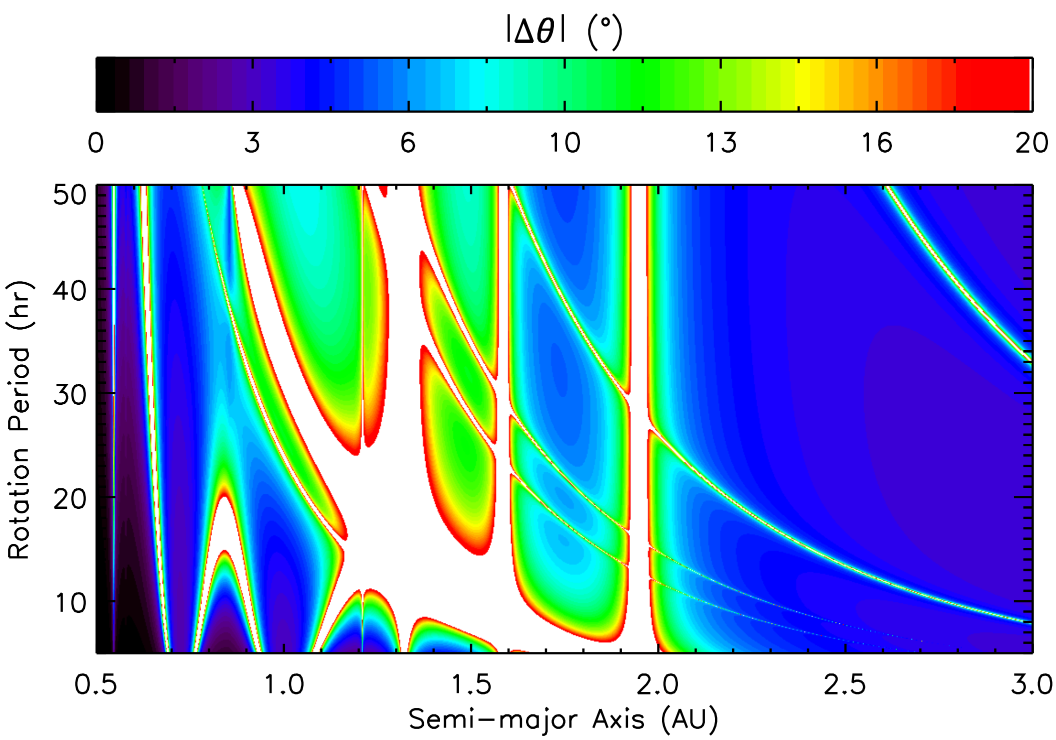

Figure 1: Obliquity variation amplitude $|\Delta\theta|$ for a test object in place of Mars as a function of rotation period and semi-major axis. White areas exceed 20^o; the linearized solution becomes increasingly inadequate at high amplitudes.